THE SEPTEMBER MEETING OF THE SAN FRANCISCO SECTION.

THE eighteenth regular meeting of the San Francisco Section of the American Mathematical Society was held at the University of California on Saturday, September 24, 1910. The following members were present:

Professor R. E. Allardice, Mr. B. A. Bernstein, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor R. L. Green, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor H. C. Moreno, Professor C. A. Noble, Professor T. M. Putnam.

Professor Blichfeldt occupied the chair. The following officers were elected for the ensuing year: Chairman, Professor Lehmer; Secretary, Professor Moreno; program committee, Professors Blichfeldt, Haskell, and Moreno.

The dates of the next two meetings were fixed as April 8, 1911, and October 28, 1911.

The following papers were read at this meeting:

(1) Professor H. F. BLICHFELDT: "On the order of linear homogeneous groups. IV."

(2) Professor D. N. LEHMER: "On the study of the general cubic by means of three involutions of rays in the plane."

Abstracts of the papers are given below:

1. In the theorems by Professor Blichfeldt on groups of linear homogeneous substitutions of determinant unity, a group of special nature is of importance, called a "self-conjugate subgroup H." It is contained in a linear group G when certain conditions are fulfilled. The theorems referred to suffer a lack of completeness due to the fact that it has not been proved that H actually contains fewer substitutions than G. In the present paper this proof has been supplied under the condition that G be primitive; besides, it is found that the number of variables n is divisible by the prime p associated with H in the theorems. An immediate result is the lowering of the known superior limit to the order of G, particularly when n is a prime.

2. Given three unrelated involutions of rays in the plane, with centers at A, B, and C, then for any point P in the plane the rays at A, B, and C that correspond in the involutions at those points to the rays PA, PB, and PC will not generally meet in a point. If, however, they do meet in a point P', then the points P and P' are said to be conjugate with respect to all three involutions. The locus of points that have conjugate points with respect to three involutions is found to be the general plane cubic. Professor Lehmer studies the curve from this point of view and connects the theory with the theory of the curve as developed by Schroeter in his Ebene Kurven dritter Ordnung (Leipzig, 1888).

C. A. NOBLE,
Secretary of the Section.

A NEW PROOF OF THE THEOREM OF WEIERSTRASS CONCERNING THE FACTORIZATION OF A POWER SERIES.

BY DR. W. D. MACMILLAN.

IN the BULLETIN for April, 1910, Bliss gives a simplified proof of the following theorem due to Weierstrass:

Let $f(y; x_1, \cdots, x_p)$ *be a convergent power series in* y *and* $x_1, \cdots, x_p$, *such that* $f(y; 0, \cdots, 0)$ *begins with a term of degree* n. *Then* $f(y; x_1, \cdots, x_p)$ *is factorable in the form*

$$f(y; x_1, \cdots, x_p) = [y^n + a_1 y^{n-1} + \cdots + a_n] \cdot g(y; x_1 \cdots, x_p),$$

where $a_1, \cdots, a_n$ *are convergent power series in* $x_1, \cdots, x_p$ *vanishing for* $x_1 = x_2 = \cdots = x_p = 0$, *and* g *is a convergent power series in* $y; x_1, \cdots, x_p$ *which has a constant term different from zero.*

Since $g(y; x_1, \cdots, x_p)$ has a constant term, we may denote its reciprocal by $\phi(y; x_1, \cdots, x_p)$ and state the theorem in the following form:

Let $f(y; x_1, \cdots, x_p)$ *be a convergent power series in* $y; x_1, \cdots, x_p$ *such that* $f(y; 0, \cdots, 0)$ *begins with a term of degree* n. *Then a convergent power series* $\phi(y; x_1, \cdots, x_p)$, *having a constant term different from zero can be found such that the product*

$f(y; x_1, \cdots, x_p)\cdot\phi(y; x_1, \cdots, x_p) = p^{(n)} = y^n + a_1 y^{n-1} + \cdots + a_n$ is a polynomial in y of degree n and $a_1, \cdots, a_n$ are convergent power series in $x_1, \cdots, x_p$, vanishing for $x_1 = x_2 = \cdots = x_p = 0$.

For the purpose of determining the coefficients of the series ϕ it will simplify the notation if we put $x_i = \xi_i \mu$. Then the series f, ϕ and $p^{(n)}$ can be written

$$
\begin{aligned}
f &= -y^n[1 - b_0] + b_1\mu + b_2\mu^2 + \cdots, \\
\phi &= c_0 + c_1\mu + c_2\mu^2 + \cdots, \\
p^{(n)} &= -y^n + p_1\mu + p_2\mu^2 + \cdots,
\end{aligned} \tag{1}
$$

where the b_k are known power series in y and are homogeneous in the ξ_i of degree k, b_0 containing terms in y only and vanishing for y equal to zero. The c_k are power series in y whose coefficients are to be determined, and the p_k are polynomials in y, also to be determined, of degree $n - 1$.

On taking the product of f and ϕ we find

$$
\begin{aligned}
f\cdot\phi = &-c_0(1-b_0)y^n + [-(1-b_0)c_1y^n + b_1c_0]\mu + [-(1-b_0)c_2y^n \\
&+ b_1c_1 + b_2c_0]\mu^2 + \cdots + \left[-(1-b_0)c_ky^n + \sum_{j=1}^{k} b_jc_{k-j}\right]\mu^k \\
&+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
= &-y^n + p_1\mu + p_2\mu^2 + \cdots + p_k\mu^k + \cdots.
\end{aligned} \tag{2}
$$

Comparing the coefficients of the various powers of μ we have

$$
\begin{aligned}
&(1 - b_0)c_0y^n = y^n, \\
&(1 - b_0)c_1y^n = b_1c_0 - p_1, \\
&(1 - b_0)c_2y^n = b_1c_1 + b_2c_0 - p_2, \\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
&(1 - b_0)c_ky^n = b_1c_{k-1} + b_2c_{k-2} + \cdots + b_kc_0 - p_k, \\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .
\end{aligned} \tag{3}
$$

These equations can be solved successively. From the first we have at once $c_0 = 1/(1 - b_0)$. From the second it is seen that, since b_1 and c_0 are known power series in y, p_1 can be chosen uniquely so that $b_1c_0 - p_1$ shall be divisible by y^n. If we suppose p_1 so chosen, we can then write $b_1c_0 - p_1 = \beta_1 y^n$ and the solution for c_1 is $c_1 = \beta_1/(1 - b_0)$. It is clear that c_1 and p_1

are homogeneous of degree 1 in the ξ_i. The third equation can be solved in the same manner, the polynomial p_2 being chosen so that the right member is divisible by y^n. Since this is possible in one and only one way, the polynomial p_2 and the power series c_2 are uniquely determined. If we suppose $c_1, \cdots, c_{k-1}$ to have been computed, then p_k can be determined uniquely so that

$$\sum_{j=1}^{k} b_j c_{k-j} - p_k$$

shall be divisible by y^n. After removing the factor y^n, c_k is determined by dividing the result by $1 - b_0$. One can thus compute as many coefficients as is desired, all the coefficients of a given order in μ being obtained in a single operation.

The proof of convergence is by dominant functions. Let us suppose that the series f converges for $|y| = 1/\sigma$, $|x_i| = 1/\rho_i$, and let us set

$$X = \frac{1}{(1-\rho_1 x_1)(1-\rho_2 x_2)\cdots(1-\rho_p x_p)} - 1.$$

Then a positive quantity M can be chosen so large that the series

$$F = -y^n + M\frac{\sigma^{n+1}y^{n+1}}{1-\sigma y} + M\frac{X}{1-\sigma y}$$

shall dominate the series f. By the above process a power series $\Phi(y; x_1, \cdots, x_p)$ and a polynomial

$$P^{(n)} = -y^n + A_1 y^{n-1} + \cdots + A_n$$

can be found such that the coefficients of Φ are all positive and greater than the moduli of the corresponding coefficients of the series ϕ; and the coefficients of the A_k, which are power series in $x_1, \cdots, x_p$ vanishing with these quantities, are all positive and greater than the moduli of the corresponding coefficients of the series a_k. Hence if the series Φ and the A_k converge so also do the series ϕ and the a_k.

Let us take

$$\Phi = \frac{1-\sigma y}{\omega - \varepsilon y},$$

where $\varepsilon = \sigma + M\sigma^{n+1}$, and ω is as yet undetermined. Then

the equation $F \cdot \Phi = P^{(n)}$ becomes

$$\left[-y^n + M\frac{\sigma^{n+1}y^{n+1}}{1-\sigma y} + M\frac{X}{1-\sigma y} \right] \frac{1-\sigma y}{\omega - sy} = -y^n + A_1 y^{n-1} + \cdots + A_n,$$

which reduces readily to

$$(4) \qquad \frac{(1-\omega)}{s}\left[\frac{y^n - \dfrac{MX}{1-\omega}}{y - \dfrac{\omega}{s}} \right] = A_1 y^{n-1} + \cdots + A_n.$$

The left member of equation (4) will be a polynomial in y of degree $n-1$ provided ω satisfies the relation

$$(5) \qquad \frac{MX}{1-\omega} = \left(\frac{\omega}{s}\right)^n.$$

On putting $\omega = 1 - z$ we get

$$(6) \qquad z = \frac{s^n MX}{(1-z)^n}.$$

Thus z is expansible as a power series in X the coefficients of which are all positive, and furthermore z vanishes with X. Hence

$$(7) \qquad \Phi = \frac{1-\sigma y}{\omega - sy} = \frac{\sigma}{s}\left[1 + \frac{M\sigma^n + z}{1 - z - sy} \right]$$

is expansible as a power series in $y, x_1, \cdots, x_p$, the coefficients of which are all positive. The polynomial $P^{(n)}$ becomes

$$P^{(n)} = -y^n + \frac{1-\omega}{s}\left[\frac{y^n - \left(\dfrac{\omega}{s}\right)^n}{y - \left(\dfrac{\omega}{s}\right)} \right] = -y^n + \frac{z}{s}y^{n-1} + \frac{z(1-z)}{s^2}y^{n-2} + \cdots + \frac{z(1-z)^{n-1}}{s^n}.$$

From equation (6) it is seen that the coefficients of this polynomial,

$$\frac{z(1-z)^k}{s^{k+1}} = \frac{s^{n-k-1}MX}{(1-z)^{n-k}} \qquad (k = 1, \cdots, n-1),$$

are expansible as power series in $x_1, \cdots, x_p$, with positive coefficients, vanishing for $x_1, = \cdots = x_p = 0$. Hence the theorem is proved.

UNIVERSITY OF CHICAGO,
August 14, 1910.

KOWALEWSKI'S DETERMINANTS.

Einführung in die Determinantentheorie einschliesslich der unendlichen und der Fredholmschen Determinanten. By GERHARD KOWALEWSKI. Leipzig, Veit & Comp., 1909. 540 pp.

A PRELIMINARY survey of the contents of this book may best be obtained by dividing it into three nearly equal parts. The first of these parts comprises Chapters I–X and deals with the pure theory together with the single application to systems of linear equations *—a subject both historically and logically so intimately connected with the theory as to be almost inseparable from it. In the second part — Chapters XI–XVI — certain applications to algebraic, analytic, and geometric problems are treated. The third part consists of Chapters XVII–XIX and deals with two extensions of the idea of determinants obtained by allowing the order of the determinant to increase indefinitely. Both the theory of these determinants (infinite determinants and Fredholm determinants) and the closely related theory of the corresponding generalizations of systems of linear equations (linear equations with an infinite number of variables and linear integral equations) are treated in these chapters. While the introduction of such subjects into a book on determinants is not wholly unprecedented — Mathews' revision of Scott's book containing a brief discussion of infinite determinants — it must still, in view of its extent, be regarded as an innovation. This feature of the book is to be welcomed and will doubtless be imitated by other writers of text-books on this subject.

* A few equations of higher degree (secular equation, etc.) which are intimately connected with the theory of determinants are also considered in this first part.

We will now take up the chapters briefly one by one:

Chapter I (5 pages) is an interesting historical introduction in which translations are given of a passage in a letter of Leibnitz containing a definition of determinants, and of the note by Cramer where his rediscovery of determinants and his rule for solving linear equations were first published.

Chapter II (17 pages) gives, after a rather elaborate preparation for the specification for the signs of the terms of a determinant, the general definition with applications to two and three rowed determinants.

In Chapter III (9 pages) the very simplest properties of determinants (interchange of two rows, multiplication of the elements of a row by a constant, etc.) are deduced; and at the close it is shown how certain of these properties characterize the determinant completely. This fact is not used in later chapters, however, as Kronecker has shown that it can be used.

Chapter IV (13 pages) is entitled Sub-Determinants and contains Laplace's development with the development according to the elements of a row as a special case. It closes with a treatment of Vandermonde's determinant, that is the determinant of the nth order whose ith row is

$$a_i^{n-1},\ a_i^{n-2},\ \cdots,\ a_i,\ 1.$$

In Chapter V (19 pages) the theory of linear algebraic equations is treated. It begins with a proof of Cramer's Rule. Then after the conception of rank has been introduced and a few simple theorems concerning it have been proved, the subject of linear dependence is discussed. Then systems of *homogeneous* equations are taken up at length; and finally the non-homogeneous case is treated.

Chapter VI (13 pages) is entitled Multiplication of Matrices and Determinants. Here, and constantly from this point on, the following notation and terminology are used: If

$$x_1,\ x_2,\ \cdots,\ x_n, \qquad y_1,\ y_2,\ \cdots,\ y_n$$

are two systems of n quantities each, the quantity

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

is called their inner product (or simply their product) and denoted by (xy). This introduction of an idea of Grassmann

is a distinctly good thing. It is, however, to be regretted that either here or at some other early stage of the book the subject were not treated in a somewhat more extensive manner by introducing the term *vector in space of n dimensions*, the conception of the sum of two vectors, the norm of a vector, the orthogonality of two vectors, etc. All of these conceptions are used freely in Chapter XVII in the case of space of an infinite number of dimensions, but before that point, where we have to deal with space of a finite number of dimensions, the term vector is not used at all, and such of these conceptions as are introduced occur in the midst of special investigations, for instance orthogonality is first explicitly mentioned on page 276 in an investigation concerning quadratic forms, although the idea (and also the conception but not the term *norm*) was really used in Chapter X. This is an illustration of the lack of complete coördination between the different parts of the book to which we shall have occasion to return later.

As a generalization and at the same time an application of this conception of the inner product we next have the product of two matrices

$$\begin{Vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{Vmatrix}, \qquad \begin{Vmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ b_{m1} & \cdots & b_{mn} \end{Vmatrix},$$

which is defined as the *determinant* of the mth order whose general element (ith row and jth column) is the product, in the sense just explained, of the ith row of the first matrix into the jth row of the second. It is proved that the product, in this sense, of two square matrices is equal to the product of their determinants (so that the theorem of multiplication of determinants is established), that when $m > n$ the product is zero, and that when $m < n$ it is the sum of the products of all corresponding m-rowed determinants in the two matrices. As an elegant application for the law of multiplication of determinants a formula for the square of Vandermonde's determinant is given which is essentially the expression for the discriminant of an algebraic equation in terms of sums of powers of the roots.

Chapter VII (33 pages) is entitled Determinants whose Elements are Minors of Another. It opens with the familiar theorems about the determinant whose elements are the cofactors

of another, and concerning the minors of this determinant. Some theorems of Sylvester of a more complicated character follow; and a treatment of bordered determinants is also included.

Chapter VIII (21 pages) on Symmetrical Determinants contains, besides a discussion of several special symmetrical determinants, an interesting proof that a symmetrical determinant of rank r contains at least one non-vanishing principal minor of order r, and a treatment of the secular equation, that is the equation obtained by equating to zero a determinant derived from a symmetrical determinant with real constant elements by adding the variable x to each element of the principal diagonal. The last section refers to a generalization of this equation which bears the same relation to a Hermitian matrix (that is to a matrix in which a_{ij} and a_{ji} are conjugate imaginary quantities) that the secular equation bears to real symmetric matrices. Some properties of Hermitian matrices are also briefly established, but the name *Hermitian*, which is freely used later in the book, is not mentioned.

In Chapter IX (27 pages) on Skew Symmetrical Determinants the following two fundamental theorems are first proved: *first* that a skew symmetric determinant of odd order is always zero, and *secondly* that a skew symmetric determinant of even order is the square of an integral rational function of its elements. This integral rational function, known as the Pfaffian, is studied, and its close analogy in theory and application to a determinant is brought out. The rank of a skew symmetric determinant is then considered; and finally a number of special determinants connected with this subject are treated.

Chapter X (19 pages) is on Orthogonal Determinants. After the fundamental properties of these determinants have been established, some special theorems of Brioschi, Siacci, and Stieltjes are proved, among which the first theorem of Siacci is especially noteworthy both for itself and for the application made of it to a simple proof of Cayley's formulæ. These formulæ, by means of which the elements of an orthogonal determinant are expressed rationally by means of $\frac{1}{2}n(n-1)$ parameters, are here proved on the supposition that the determinant has the value $+1$ and that the determinant obtained by adding 1 to each element of the principal diagonal is not zero. The formulæ are worked out in detail for determinants of the second and third orders.

A footnote at the beginning of this chapter tells us that the

study of orthogonal determinants shall here be restricted to the case in which all the elements are real. This seems at first quite incomprehensible since, except at one or two points, where the restriction might easily have been explicitly made, the results and methods of the chapter apply equally well to the case of complex elements. The reason for the restriction to real elements becomes evident only at a much later point in the book (page 282) where, in the course of a special investigation on Hermitian forms, the word *orthogonal* is applied to the case of complex elements when the quantity

$$a_{i1}\bar{a}_{j1} + a_{i2}\bar{a}_{j2} + \cdots + a_{in}\bar{a}_{jn}$$

has the value 1 or 0 according as i and j are equal to or different from each other, — the a's being the elements and dashes being used to denote the conjugate complex quantity. While such a departure from well established usage in terminology can hardly be countenanced, it is certain that we have here a conception of importance * which under some other name is well worthy of place in a book on determinants. It is only to be regretted that it was not treated in a somewhat systematic manner in the chapter on orthogonal determinants.

We pass now to what we have called the second part of the book, — Applications.

Chapter XI (32 pages) on Resultants and Discriminants deals exclusively with binary forms. Sylvester's dialytic method (supplemented so as to be complete), Bézout's method, the relation of resultant and discriminant to the linear factors of the forms, finally their invariant character together with a few words about invariants of binary forms in general are some of the main subjects treated. The treatment at some points, for instance in §§ 78, 79, is remarkably elegant and lucid.

Chapter XII (43 pages) on Linear and Quadratic Forms opens with a consideration of the rank of a system of linear forms and, in case the number of forms is equal to the number of variables, their resultant regarded as invariants with regard to linear transformations. Then comes a brief treatment of bilinear forms in which, for the first time, that product of two matrices is considered which is itself a matrix. The remainder

* Cf. Frobenius, *Crelle*, vol. 95 (1883), p. 267; E. Study, *Math. Ann.*, vol. 60 (1905), p. 321, and I. Schur, *Math. Ann.*, vol. 66 (1909), p. 489, where references to some further literature will be found.

of the chapter is devoted to quadratic forms, their reduction to sums of squares, the law of inertia and the classification of real quadratic forms, the adjoint form, the rational invariants of pairs of quadratic forms, etc. Here, with slight exceptions, both the facts proved and the methods of proof are identical with those contained in the reviewer's Introduction to Higher Algebra.

Chapter XIII (36 pages) is on Elementary Divisors, and gives, in very brief space, the fundamental theorems on the equivalence of pairs of bilinear or quadratic forms. The method used, which is due to Stickelberger, depends, as did Weierstrass's original method, on a reduction to normal form, and like that method it involves unnecessary irrationalities. It depends on analytic tricks for which no motive is apparent, though the author has met this last objection to some slight extent by a similar, though much simpler discussion for a single quadratic form in § 106. Its compensating advantage, apart from its brevity, is that, like Weierstrass's original method, it gives a practicable method of approach to the subject of real quadratic forms, a subject in which, however, only a very special case is here considered.

The last part of this chapter contains an interesting exposition of the real *orthogonal* reduction of a single real quadratic form to a form in which only square terms enter, together with a generalization to the case of Hermitian forms. This work does not involve the theory of elementary divisors in any way, so that the place which has been given it, at the end of the chapter on elementary divisors, is unfortunate as many readers without the time or inclination to make themselves familiar with this theory will be likely to miss altogether this elegant treatment.

Chapter XIV (30 pages) on Functional Determinants opens, after the conception of the Jacobian has been defined, with the remark that the Jacobian

$$\frac{d(u_1, \cdots, u_n)}{d(x_1, \cdots, x_n)}$$

can be written as the quotient

$$\begin{vmatrix} d_1u_1 & d_1u_2 & \cdots & d_1u_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ d_nu_1 & d_nu_2 & \cdots & d_nu_n \end{vmatrix} : \begin{vmatrix} d_1x_1 & d_1x_2 & \cdots & d_1x_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ d_nx_1 & d_nx_2 & \cdots & d_nx_n \end{vmatrix}$$

where $d_i u_j$ denotes the differential of u_j which corresponds to the increments

$$d_i x_1,\ d_i x_2,\ \cdots,\ d_i x_n,$$

and where the increments of the x's are chosen arbitrarily but so that their determinant is not zero. It is then shown that, under suitable restrictions, the Jacobian can be obtained as the limit of a similar ratio where the differentials of the u's are replaced by increments. It would have been interesting for the reader to have been told whether this result is of any importance, and if so why.

There follows the law of multiplication for functional matrices and determinants. The transformation

$$\begin{aligned}
\xi_1 &= u_1(x_1, \cdots, x_n),\\
\xi_2 &= u_2(x_1, \cdots, x_n),\\
&\cdots\cdots\cdots\\
&\cdots\cdots\cdots\\
\xi_n &= u_n(x_1, \cdots, x_n),
\end{aligned}$$

when the Jacobian of the u's is not zero, is then studied for the neighborhood of individual points, the work being carried through in detail for the case $n = 2$. Then comes a treatment of the effect of the rank of the functional matrix on the number of independent functions in the neighborhood of a non-singular point. Finally inverse functions, implicit functions, and the minors of the Jacobian are taken up.

Chapter XV (17 pages) on Wronskian and Gramian Determinants is one of the most novel and timely in the book. If $f_1(x), \cdots, f_n(x)$ are continuous (or more generally if these functions and their squares are integrable), but not necessarily real, throughout the interval $a \leqq x \leqq b$, we understand by their Gramian the determinant of the nth order in which the term in the ith row and jth column is

$$\int_a^b f_i(x)\bar{f}_j(x)dx,$$

the dash indicating the conjugate complex quantity. The vanishing of this determinant is a necessary and sufficient condition

that $f_1, \cdots, f_n$ be linearly dependent.* If $f_1, \cdots, f_n$ are linearly independent, their Gramian is always positive. On the other hand, if

$$a_{i1}, a_{i2}, \cdots, a_{ik} \qquad (i = 1,2, \cdots, n)$$

are n sets of k constants each, we understand by their Gramian the determinant of the nth order in which the element in the ith row and jth column is

$$a_{i1}\bar{a}_{j1} + a_{i2}\bar{a}_{j2} + \cdots + a_{ik}\bar{a}_{jk}.$$

Here again the vanishing of this determinant is a necessary and sufficient condition for the linear dependence of the sets of constants; and if these sets are linearly independent, their Gramian is positive. We have here a striking analogy, but unfortunately no attempt is made in the book under review to explain *why* this analogy exists. Such an explanation would not have been difficult, for instance one may regard the case last mentioned as a special case of the earlier one by letting $a = 0$, $b = k$ and

$$f_i(x) = a_{ij} \quad \text{when} \quad j - 1 < x < j$$
$$(i = 1, 2, \cdots, n\,; j = 1, 2, \cdots, k).$$

Or one may take the more general point of view of considering n sets of k *functions* each

$$f_{i1}(x), f_{i2}(x), \cdots, f_{ik}(x) \qquad (i = 1, 2, \cdots, n)$$

continuous throughout the interval $a \leqq x \leqq b$, or, more generally, such that these functions and their squares are integrable throughout this interval. By their Gramian we then understand the determinant of the nth order in which the element in the ith row and jth column is

$$\int_a^b [f_{i1}(x)\bar{f}_{j1}(x) + \cdots + f_{ik}(x)\bar{f}_{jk}(x)]dx,$$

* If $f_1, \cdots, f_n$ are not assumed to be continuous, we understand here by linear dependence throughout the interval $a \leqq x \leqq b$ the existence of n constants $c_1, \cdots, c_n$ not all zero and such that

$$c_1 f_1 + \cdots + c_n f_n$$

vanishes at every point of this interval except at the points of a set of content zero.

and here again this Gramian will always be positive except when the n sets of functions are linearly dependent, in which case it vanishes. This conception clearly includes the two above mentioned as special cases.

However much the chapter under consideration is to be welcomed, it is to be regretted that it had not been made more complete. This might have been effected in part by collecting here various discussions which are scattered through the later chapters and which really refer to this subject; for instance the proof of Hadamard's inequality for the absolute value of a determinant on pages 458–460. This inequality is simply a corollary to the theorem that a Gramian (in any of the senses above mentioned) can never exceed the product of the elements of its principal diagonal.* Again the relation of Gramians to definite Hermitian forms is not considered in this chapter.

One section of this chapter is devoted to the relation of Wronskians to the subject of linear dependence, but after three and a half pages of this discussion the reader is left without the knowledge of the most important, and one of the most easily proved theorems in this subject, namely that if n functions are analytic, the identical vanishing of their Wronskian is a sufficient condition for linear dependence.

In comparing the Wronskian with the Gramian test for linear dependence the important fact is pointed out that while the former test requires, in the case of n functions, the existence of the first $n-1$ derivatives, for the latter the continuity of the functions is sufficient. The even more important fact that the Gramian test applies to functions in any number of variables is not mentioned.

Chapter XVI (32 pages) is entitled: Some Geometrical Applications of Determinants. More than half of it is devoted to applications in the domain of elementary analytic geometry: The area of a triangle in terms of the coördinates of its vertices and in terms of the equations of its sides; the product of the areas of two triangles in a plane in terms of the distances between their six vertices; a similar result in case the two triangles are inscribed in a circle of radius r; the characteristic relation between the distances of four points on a circle; finally the generalization of all of these results to space. Then come two

* If none of the functions (or systems of functions) are identically zero, it reaches this upper limit when and only when the functions (or systems of functions) are mutually orthogonal.

sections on affine transformations in the plane, with special reference to the fact that all areas are changed in the same ratio. Finally the geometrical interpretation of the Jacobian is discussed of which the property of affine transformations just mentioned is a special case, the work being here again confined to the neighborhood of a non-singular point.

* * *

We pass now to the third part of the book.

Chapter XVII (86 pages): Determinants of Infinitely High Order, might well have been split into two chapters. It should at least have been clearly indicated that it consists of two parts wholly independent of each other. The first part, somewhat the shorter of the two, closes on page 407. If we have a doubly infinite array of quantities

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

and if we form from them the determinant of the nth order

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix},$$

it may happen that this determinant has a finite limit when $n = \infty$. This limit is then called a determinant of infinitely high order and is denoted by putting vertical bars on the two sides of the infinite array above. Our author considers primarily *normal* determinants, that is determinants for which the double *series*

$$\sum_{i,j=1}^{\infty} c_{ij}$$

converges where

$$c_{ii} = |a_{ii} - 1|, \quad c_{ij} = |a_{ij}| \qquad (i \neq j).$$

Such determinants are necessarily convergent as are also their

minors and all determinants obtained from them by replacing a finite number of their rows or columns by quantities which are in absolute value less than some constant.

If in the infinite system of linear equations with an infinite number of variables

$$a_{11}x_1 + a_{12}x_2 + \cdots = b_1,$$
$$a_{21}x_1 + a_{22}x_2 + \cdots = b_2,$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$

the a_{ij}'s are such that the infinite determinant formed from them is a normal determinant, and the b's do not exceed in absolute value some constant, it can be shown that normal determinants and their minors can be used for the solution of the system precisely as finite determinants can be used for the solution of finite systems of linear equations with a finite number of variables. This, together with the extension of numerous properties of finite determinants to infinite normal determinants, forms the contents of this first part of the chapter.

The second part contains an exposition of E. Schmidt's theory of systems of linear equations, and *does not deal with infinite determinants at all.* It is true that determinants occur here whose order n increases indefinitely, but these determinants do not in general themselves approach finite limits when $n=\infty$ but only the ratio of two such determinants.* The subject has been made more attractive by the free use of the conception and language of vectors in space of an infinite number of dimensions. One misses, however, any statement of the relation of this method of Schmidt to the method of infinite determinants explained in the earlier part of the chapter. The relations would seem to be, in part, these:

1. Both methods deal with the solution of the infinite system of equations referred to above, and neither method treats the problem in all its generality.

2. In the method of infinite determinants it is assumed that the double series $\sum c_{ij}$ converges, while in Schmidt's method not even the convergence of the individual rows of this series,

* The determinants themselves can be made to converge by dividing the equations of the system we wish to solve by constants such as to make the series of the squares of the coefficients of the individual equations have values less than or equal to 1.

or, what amounts to the same thing, the convergence of the series

$$\sum_{j=1}^{\infty} |a_{ij}| \qquad (i = 1, 2, \cdots)$$

is demanded, but merely the convergence of the series

$$\sum_{j=1}^{\infty} |a_{ij}|^2 \qquad (i = 1, 2, \cdots).*$$

3. The method of infinite determinants yields, in the cases to which it is applicable, every solution $(x_1, x_2, \cdots)$ which is such that all the x_i's are in absolute value less than some constant, while Schmidt's method yields only those solutions $(x_1, x_2, \cdots)$ for which the series $\sum |x_i|^2$ converges.

4. When we use infinite determinants, we obtain the solution of the infinite system † as the limit for $n = \infty$ of the solution of the finite system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n, \end{aligned}$$

that is we allow the number of equations and the number of unknowns to increase *simultaneously* always remaining equal. In Schmidt's method we start from the system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1k}x_k &= b_1, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ a_{n1}x_1 + \cdots + a_{nk}x_k &= b_n \end{aligned}$$

and first, keeping n fixed, pass to the limit $k = \infty$;‡ only after this has been done do we allow n to become infinite. In order to be able to carry this method through, new formulae must be obtained for the solution of the system of equations last written when $k > n$.

Some such explanation as this of the scope of the two methods and their relation to one another should have been given in the

* We pass over here the less easily stated conditions on the b's.

† At least in case the determinant of the system is not zero.

‡ This first passage to a limit is, in practice, avoided by treating *directly* the case $k = \infty$.

chapter under discussion. Instead of this, the two methods are merely placed side by side without comment. Moreover in the latter part of the chapter we miss any clear statements of the results obtained, so that it is almost impossible to find out precisely what Schmidt's method accomplishes without reading the greater part of the forty odd pages here devoted to it.

Chapter XVIII (50 pages) on Linear Integral Equations opens with a discussion of Fredholm's determinant. This, like the infinite determinant, is the limit of a determinant of the nth order as n becomes infinite; but while, in the case of the infinite determinant, the element in the ith row and jth column of the determinant of the nth order does not change with n when once n is greater than both i and j (just as in the case of an infinite series, which is the limit of the sum of n terms, these terms do not change with n), in the case of Fredholm's determinant the elements approach zero, or, in the case of the elements of the principal diagonal, one, as n becomes infinite (just as in the case of a definite integral, which is also the limit of a sum of n terms, each term approaches zero as n becomes infinite). To make this rough statement more precise, consider a function $f(x, y)$ continuous throughout the square

$$S \qquad 0 \leqq x \leqq 1, \quad 0 \leqq y \leqq 1.$$

Divide S into n^2 equal squares by lines parallel to the axes of x and y and denote the coordinates of the centres of these squares by

$$(x_i, y_j) \qquad (i, j = 1, 2, \cdots, n),$$

where the notation is such that both x_i and y_j increase as the subscript increases. Then, as n becomes infinite, the determinant

$$D_n = \begin{vmatrix} 1 + \frac{f(x_1, y_1)}{n} & \frac{f(x_1, y_2)}{n} & \cdots & \frac{f(x_1, y_n)}{n} \\ \frac{f(x_2, y_1)}{n} & 1 + \frac{f(x_2, y_2)}{n} & \cdots & \frac{f(x_2, y_n)}{n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{f(x_n, y_1)}{n} & \frac{f(x_n, y_2)}{n} & \cdots & 1 + \frac{f(x_n, y_n)}{n} \end{vmatrix}$$

approaches a finite limit called the Fredholm determinant of the function f for the region S.

About half of the chapter under consideration is devoted to the theory of these Fredholm determinants; their minors and relations between them, the product of two Fredholm determinants, etc., being discussed. The *results* are due to Fredholm; the *method*, that of passing to the limit from the algebraic case, was used by Fredholm as a heuristic one for discovering the facts. His method of proof, however, was a different and more elegant one. As a rigorous method of proof, this method was first used by Hilbert for establishing some of Fredholm's results, along with certain other results with which we are not here concerned. In other cases (the minors of higher orders) the chapter now under consideration seems to give the first exposition of this method of proof.

The other half of the chapter is devoted to the application of Fredholm's determinants to the integral equation of the second kind

$$\phi(x) + \int_0^1 f(x, y)\phi(y)dy = \psi(x),$$

where f and ψ are known functions, ϕ the function to be determined. Here the method used is that of Fredholm not of Hilbert, that is the integral equation is treated directly, not as the limit of a system of linear algebraic equations. The analogy with the case of linear algebraic equations is frequently mentioned and sometimes elaborated at some length, but we fancy this analogy will seem pretty obscure unless the reader is fortunate enough to notice the note on page 544, where the method of obtaining the integral equation as the limit of the algebraic system is explained and falsely attributed to Hilbert. It was known to Fredholm, and had been explained by Volterra in 1896.

Here, as in Chapter XV, it is unfortunate that no mention is made of the fact that methods and results apply without change to functions in any number of variables.

Chapter XIX (35 pages) is entitled: Hilbert's Characteristic Functions of a Real Symmetric Kernel, and gives, with certain interesting modifications of detail, some of the more important parts of E. Schmidt's dissertation (*Mathematische Annalen*, volume 63). It deals with the above written integral equation, chiefly in the case in which f is a symmetric function of (x, y), with special attention to the case in which ψ vanishes identically. While these developments of Schmidt are of the great-

est importance and elegance, it is hard to see why they should be inserted into a book on determinants. More germane would seem to be Hilbert's theory of quadratic forms with an infinite number of variables, an exposition of which would have been, for other reasons also, most desirable.

* * *

Properly regarded, the theory of determinants is hardly more than a shadow of the theory of matrices, and just so far as one attempts to ignore this fact does the subject become an artificial one or, at best, a tool for other investigations. Every book on determinants must be dominated more or less by the conception of the matrix, which first appears as an inanimate object, a mere rectangular array of quantities from which determinants, whose vanishing or non-vanishing are observed or whose values are used, are cut out. Later, however, the matrix appears as an animated being capable of combination by addition and at least two kinds of multiplication with its kind, and it is in this aspect that the conception becomes a highly fruitful one. In the book under review the matrix as an inanimate object is freely used, and the less important kind of multiplication, according to which the product of two matrices is a determinant (cf. what was said in connection with Chapter VI) is adequately treated. That two matrices can be added together or that a matrix can be multiplied by an ordinary quantity (scalar) seems to be nowhere even mentioned. The extremely important kind of multiplication according to which the product of two square matrices of the nth order is another matrix of the nth order is not taken up until the section on Bilinear Forms in Chapter XII, that is, in what we have called the second part of the book, after the *theory* of determinants of finite order has been completed. Thus it will be seen that the idea of the matrix is used somewhat reluctantly. That the book nevertheless remains a good book is due largely to the fact that in many places where the word *determinant* is used a matrix is meant. A determinant is quite properly defined (page 19, near the top) as a *quantity* obtained by combining by the well-known rule the elements of a square matrix. On page 24 we have the theorem that a determinant is unchanged by an interchange of rows with columns. Later (page 161) comes the theorem that an orthogonal determinant remains orthogonal if rows are interchanged with columns. What a triviality if the earlier statement that every determinant remains unchanged was true!

The fact of course is that the theorem about orthogonal determinants is meant to refer not to a determinant, but to an orthogonal matrix. Indeed the whole chapter on orthogonal determinants is really a chapter on orthogonal matrices; the only fact proved, and the only non-trivial fact which can be proved about an orthogonal determinant (i. e., the determinant of an orthogonal matrix) being that it is equal to $+1$ or -1. All that we can mean by saying that a determinant (i. e., a quantity) is orthogonal is that an orthogonal matrix exists of which it is the determinant; and this is clearly true when and only when the determinant is $+1$ or -1. Consequently, to take one more example, the theorem that the product of two orthogonal determinants is an orthogonal determinant is a triviality — it merely says that if $a = \pm 1$ and $b = \pm 1$, then $ab = \pm 1$. What is *meant* is, of course, that the product of two orthogonal *matrices* is an orthogonal *matrix*, the kind of product meant being that which is defined only fifty pages later.

The advantage of the free use of the word and conception *matrix* would be not merely to avoid calling two different things by the same name, perhaps even in a single sentence, — a most pernicious thing in mathematics — but also to permit much briefer and more transparent formulations of many proofs. For instance, it follows at once from the definition that a matrix A is orthogonal when and only when

$$A'A = I,$$

where A' is the conjugate (transposed) of A and I is the unit matrix

$$I = \begin{Vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{Vmatrix}.$$

It follows that

$$A' = A^{-1}$$

and this is merely a more compact and elegant way of stating Theorem 43 on page 160. Since a matrix is always commuta-

tive with its reciprocal, we see that

$$AA' = I$$

and consequently A' is orthogonal. (Theorem 44, page 161.)

If A and B are both orthogonal

$$A' = A^{-1}, \quad BB' = I.$$

Multiplying these equations together gives

$$A'BB' = A^{-1}.$$

Consequently

$$A'BB'A = I.$$

But, $B'A$ being the conjugate of $A'B$, the orthogonality of $A'B$ is thus proved. (Theorem 45, page 161.)

The introduction of proofs of this character would presuppose a treatment of the fundamental facts in the algebra of matrices. The few pages necessary for this purpose would be well spent, as some of the most important papers published at present on determinants are written in part in the language of matrices.*

One of the strong features of the book is the admirable selection of subjects; for though, as we have pointed out, there are a few things which one would like to have seen added, there is very little which one would be willing to see omitted. Open the book where you will and you find yourself in the midst of a live question having vital connections at the present day with other live branches of mathematics. This is no small praise when one recalls the possibilities for barren formalism which the subject of determinants presents. The choice not only of subjects but of the special aspects of these subjects to be considered and of the individual theorems to be proved has been guided by a true sense of values, not by the mere love of formal developments.

It is therefore somewhat surprising that in the selection of methods of proof a tendency to give the preference to formal manipulations is frequently apparent. Someone has said: "C'est le sentiment de se mouvoir dans l'identique qui fait la joie de l'algébriste" — a statement which would be correct if the word *formaliste* were put in place of *algébriste*. The algebraist has no less pleasure in taking large steps where the

*Cf., for instance, the paper by I. Schur already cited.

feeling of identity is lost than has the analyst or the geometer; and indeed the criticism just made that many proofs in the book under review are of too purely formal a character is closely connected with the further fact that the author seems to make an effort to use as little algebra as possible. The conception of reducibility is nowhere used, and it is only in a brief note at the end of the volume (page 542) that the irreducibility of the general determinant is mentioned. Except in the case of the binary form, no reference is made to the fact that every polynomial can be resolved into irreducible factors in one and essentially only one way. Even such a fundamental fact as that a polynomial vanishes identically when and only when all of its coefficients are zero is not assumed to be known but is *proved* when, in the chapter on Elementary Divisors (page 258), it is found necessary.* Methods so thoroughly germane to the theory of determinants as are the elementary agebraic theorems just mentioned are avoided at the expense of simplicity and brevity. For instance, in the treatment of Vandermonde's determinant in Chapter IV, a formal proof covering two pages is given. This may perhaps be justified on the ground that at this early stage practice in applying the rules for manipulating determinants is desirable; but a satisfactory reason can hardly be given for the omission of the brief and lucid proof which says that the determinant, being a polynomial in $a_1, \cdots, a_n$ which vanishes when any two a's are equal, must have each of the differences $a_i - a_j$ as a factor. The other factor, into which the product of these differences is multiplied, which is clearly a constant, can then be easily determined.†

Another good illustration of the tendency in question is to be found in the proof that the determinant of a skew symmetric matrix of even order is the square of a polynomial in the elements of the matrix. Two proofs of this theorem are given, the first of which covers a page and a half, and the second (which is of a purely formal character involving an examination of the individual terms of the determinant) three pages. The theorem may be proved by mathematical induction, a method also used in the first of the two proofs just mentioned, as follows:

* Contrast with this the free use made, without explanation, of uniform convergence, the integration of series term by term, and various matters relating to the differential and integral calculus.

† We must use here the algebraic theorem (or at least an easily proved special case of it) which says that if a first polynomial vanishes whenever a second irreducible one does, the second is a factor of the first.

Denote the determinant of order $2n$ by D, its cofactors by A_{ij} and the determinant obtained by striking out its last two rows and columns by S. Then

$$SD = A_{2n-1,\,2n-1}A_{2n,\,2n} - A_{2n,\,2n-1}A_{2n-1,\,2n}.$$

But since $A_{2n-1,\,2n-1}$ and $A_{2n,\,2n}$ are skew symmetric determinants of odd order and therefore vanish, while

$$A_{2n,\,2n-1} = -A_{2n-1,\,2n},$$

the above formula reduces to

$$SD = A^2_{2n,\,2n-1}.$$

Consequently since S, being a skew symmetric determinant of order $2n-2$, is by hypothesis the square of a polynomial in the elements, it follows, by an application of the theorem that a polynomial can be resolved into irreducible factors in essentially only one way, that the same is true of D.

The obvious reason for the second and longer proof is to introduce the subject of Pfaffians. This is, however, quite unnecessary. We may define the Pfaffian of a skew symmetric matrix of order $2n$ as the square root of its determinant, the sign being so determined that the Pfaffian reduces to $+1$ when

$$a_{12} = -a_{21} = 1,\ a_{34} = -a_{43} = 1,\ \cdots,\ a_{2n-1,\,2n} = -a_{2n,\,2n-1} = 1$$

while all the other a's are zero. This Pfaffian is then, by the theorem just proved, a polynomial of degree n in the a_{ij}'s the forms and signs of whose terms can be determined with ease by methods which make far less demand on the concentrated attention of the reader than is called for in the section under consideration, and which also have the advantage of much greater brevity.

We have already indicated by various examples that the relations of different parts of the book to each other are not always clearly brought out. A minor, though not unimportant illustration of this same weakness is to be found in matters of terminology. The definition on page 78 of the reciprocal determinant does not tally with the definition on page 241 of the reciprocal quadratic form. The definition on page 49 of the rank of a finite determinant (or more accurately matrix) does not tally with the definition on page 399 of the rank of an

infinite determinant, nor with the definition on page 484 of the rank of a Fredholm determinant. Connected with a matrix of order n are two integers, its *rank* and its *nullity*.* The sum of these two numbers is n, so that we may dispense with either one of them, and most authors use the rank alone. It is, however, sometimes convenient to use the nullity, and it is the generalization of this latter conception, not of the rank, with which we have to deal in the case of infinite determinants and of Fredholm determinants.

A more detailed table of contents containing section as well as chapter headings, and a more extensive index would have been of much assistance to users of the book.

Many will be inclined to criticise the scantiness of the bibliographical references placed, along with a few other notes, at the end of the volume. It must be remembered, however, that the collection of *reliable* information of this sort for such a widely ramified and, in the main, old field as that here treated is a very serious undertaking. An extensive and reliable bibliography would be, without doubt, a very welcome supplement, but it seems to be asking too much of the author of a book of this character to demand of him that he undertake such a task; the difficulties in producing good mathematical treatises are quite formidable enough without that.

We have said that the choice of subjects treated, both in the large and in detail, is admirable. The book has seemed to us at some points less satisfactory in the *methods* of proof selected, but these methods having once been chosen, the detailed exposition of the proof is always good and frequently a model of clearness and conciseness; it is evident that we have to deal with a writer whose expository powers are of the first order.

Last but not least, the standard of accuracy attained is remarkably high. In the sections which we have examined with care, and these constitute no inconsiderable portion of the book, the only error we have found is the one already mentioned of using the word *determinant* frequently in the sense of *matrix*, and this error is so sanctioned by universal usage that it must, for the present, be regarded rather as a defect of the age than of the individual. Only an author who has himself seriously tried to eliminate errors from his writings and is not satisfied, when a mistake is discovered, to say that he

* The word *Defekt* has been used here in German.

wishes to be understood in some other sense, which his words do not admit (in other words to *crawl* if we may be permitted to use an expressive vulgarism) can appreciate the labor involved in the production of a text-book with this high record of correctness. We can recommend the book, in spite of some shortcomings, as one from which few readers can fail to get much valuable information without undue effort.

MAXIME BÔCHER.

DIFFERENTIAL INVARIANTS.

Invariants of Quadratic Differential Forms. By J. EDMUND WRIGHT. Cambridge Tract No. 9. Cambridge University Press, 1908. viii + 90 pp.

IN these days, when the number of papers in mathematics published each year is almost without limit and the ramifications are no less perplexing in their variety, one is delighted to find here and there a digest of the work in a particular field. These are the pleasures which the Cambridge Tracts hold in store for us, and we owe a debt of gratitude to our fellow-workers who are willing to pause in their researches to give us a panoramic view of their field and thus to turn over to us in nut-shell form the products of their searchings in the works of their colaborers. The author of the tract before us felt that this was its mission and he seems to have attained his hopes.

In his introduction he leads the reader into the domain of his subject by showing him an invariant and then defining it. The example taken is naturally enough the Gaussian curvature of a surface whose linear element

$$ds^2 = dx^2 + dy^2 + dz^2,$$

expressed in terms of two independent parameters u and v, is written

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

The idea of differential parameter is also suggested at this time. The reader is then acquainted with the magnitude of the field of inquiry and of the two kinds of problems: (i) The determination of all invariants of one or more differential quadratic forms and their relations; (ii) The geometrical and mechanical interpretation of these invariants.

In Chapter I the author gives a brief historical sketch of the development of the theory, indicating in particular the lines of attack and the leaders in these subfields. Here one finds almost all of the references to the memoirs in which the subject-matter is to be found. Subsequently the reader is inclined to feel that it would have been more helpful had the references been given along with the more or less extended development of the matter as set forth in the succeeding chapters. In these chapters, because of the limited amount of space, the treatment is necessarily brief and so it is desirable that the hurried reader should be able to dip at times into the original memoirs.

Chapter II opens with a short discussion of the problem of finding the necessary and sufficient conditions that two given surfaces be applicable to one another as an introduction to the general problem proposed by Christoffel * which may be stated as follows:

Given two differential quadratic forms in n variables

$$(1) \qquad F = \sum_{i,\kappa} a_{i\kappa} dx_i dx_\kappa, \quad F' = \sum_{i,\kappa} a'_{i\kappa} dy_i dy_\kappa,$$

where $a_{i\kappa}$ and $a'_{i\kappa}$ are functions of the x's and y's respectively; what are the necessary and sufficient conditions that a transformation of the variables exist, say

$$(2) \qquad x_i = \phi_i(y_1, \cdots, y_n),$$

such that F is transformable into F'?

This is equivalent to the determination of the conditions under which the system of partial differential equations

$$(3) \qquad a'_{i\kappa} = \sum_{r,s} a_{rs} \frac{\partial x_r}{\partial y_i} \frac{\partial x_s}{\partial y_\kappa}$$

admits a solution. This question is reduced by Christoffel to an algebraic problem.

If equations (3) be differentiated with respect to y_l, where $l = 1, \cdots, n$, and the resulting equations be solved for the second derivatives, the conditions that these equations be consistent are expressible in the form

$$(4) \qquad (\alpha_1\alpha_2\alpha_3\alpha_4)' = \sum_{i_1 i_2 i_3 i_4} (i_1 i_2 i_3 i_4) \frac{\partial x_{i_1}}{\partial y_{\alpha_1}} \frac{\partial x_{i_2}}{\partial y_{\alpha_2}} \frac{\partial x_{i_3}}{\partial y_{\alpha_3}} \frac{\partial x_{i_4}}{\partial y_{\alpha_4}},$$

* "Ueber die Transformation der homogenen Differentialausdrücke des zweiten Grades," *Crelle*, vol. 70 (1870).

where $(i_1i_2i_3i_4)$ denotes a certain function of the quantities a_{rs} and their first and second derivatives, and $(\alpha_1\alpha_2\alpha_3\alpha_4)'$ a similar function in terms of the functions a'_{rs}. These functions $(i_1i_2i_3i_4)$ were used first by Riemann in the Commentatio Mathematica and so Ricci has called them the *Riemann symbols.*

Similarly to (3) equations (4) are the conditions of equivalence of a form

$$G_4 = \sum_{i_1i_2i_3i_4} (i_1i_2i_3i_4)d^{(1)}x_{i_1}d^{(2)}x_{i_2}d^{(3)}x_{i_3}d^{(4)}x_{i_4}, \tag{5}$$

and an analogous form G'_4 in the y's. Here we take four different sets of differentials, since there are certain linear relations between Riemann symbols, in consequence of which G_4 vanishes identically when the differentials are the same. Thus equations (4) are the conditions for the equivalence of two *quadrilinear* forms.

If equations (4) be differentiated with respect to y_a and if we make use of five index symbols defined by

$$\begin{aligned}(i_1i_2i_3i_4i) &= \frac{\partial}{\partial x_i}(i_1i_2i_3i_4) \\ &\quad - \sum_\lambda \left[\left\{ \begin{matrix} ii_1 \\ \lambda \end{matrix} \right\} (\lambda i_2i_3i_4) + \left\{ \begin{matrix} ii_2 \\ \lambda \end{matrix} \right\} (i_1\lambda i_3i_4) + \cdots \right],\end{aligned} \tag{6}$$

where $\left\{ \begin{matrix} ij \\ k \end{matrix} \right\}$ is the usual Christoffel symbol, we obtain equations

$$(\alpha_1\alpha_2\alpha_3\alpha_4\alpha)' = \sum_{i_1i_2i_3i_4i} (i_1i_2i_3i_4i) \frac{\partial x_{i_1}}{\partial y_{a_1}} \cdots \frac{\partial x_i}{\partial y_a}, \tag{7}$$

which evidently are the conditions for the equivalence of two quinquilinear forms G_5 and G'_5.

Continuing by means of a process which is an immediate generalization of (6), we obtain a series of equations analogous to (7) and thus a sequence of covariant forms $G_4, G_5, \cdots, G_\mu$.

The author gives few details in connection with the statement of the foregoing results, but he develops at some length the proof of the following fundamental theorem of Christoffel:

The necessary and sufficient conditions in order that it shall be possible to transform a quadratic form F into another quadratic form F' are that the equations in the variables $x, y, \partial x/\partial y$ derived from the equivalence of two sequences

$$F, G_4, G_5, \cdots, G_\mu \quad \text{and} \quad F', G'_4, \cdots, G'_\mu$$

shall be algebraically compatible.

Wright shows that two cases are to be considered:

(i) When an order q can be determined such that the equations

$$F = F', \quad G_4 = G'_4, \quad \cdots, \quad G_{q-1} = G'_{q-1}$$

determine the quantities x and $\partial x/\partial y$ as functions of the y's and these values make G_q and G'_q an identity.

(ii) When the above equations yield only $p(< n^2 + n)$ independent equations, and if any set of solutions of these p equations satisfy identically $G_q = G'_q$.

When the conditions of the first case are satisfied the problem is algebraic, but in the second case certain of the quantities left undetermined by the algebraic conditions must satisfy partial differential equations of the first order. From the algebraic theory it follows that the algebraic invariants of the two sets of multilinear forms must be equal. And the equations of transformation are given by equating n independent absolute invariants. Hence it is necessary that the sequence of forms F, G_4, G_5, $\cdots$ be extended to such a point that there shall be n absolute simultaneous invariants.

In particular the invariants I, I_1, $\cdots$ of the algebraic forms F, G_4, $\cdots$ are a complete system of relative differential invariants of the form F, and if under any transformation I becomes κI, then κ is some power of the Jacobian of the transformation.

The remainder of Chapter II is devoted to a discussion of the "absolute differential calculus" which Ricci and Levi-Civita* have developed from the fundamental principles discovered by Christoffel.

Equations (3) may be looked upon as defining a transformation of the functions a_{r_1} simultaneously with the transformation (2) of the variables. The same may be said of equations (4) and (7). All of these equations are of the form

$$(8) \qquad Y_{r_1 \cdots r_m} = \sum_{s_1 \cdots s_m} X_{s_1 \cdots s_m} \frac{\partial x_{s_1}}{\partial y_{r_1}} \cdots \frac{\partial x_{s_m}}{\partial y_{r_m}}.$$

Functions $X_{s_1 \cdots s_m}$ whose transformation equations are of this sort are said to form a *covariant system* of order m. In similar manner, the functions $X^{(s_1 \cdots s_m)}$ whose transformation equations

* *Math. Annalen*, vol. 54, pp. 125–201.

are of the form

$$Y^{(r_1 \cdots r_m)} = \sum_{s_1 \cdots s_m} X^{(s_1 \cdots s_m)} \frac{\partial y_{r_1}}{\partial x_{s_1}} \cdots \frac{\partial y_{r_m}}{\partial x_{s_m}}$$

form a *contravariant system* of order m. As an example, the coefficients $a^{(rs)}$ of the quadratic form reciprocal to F form a contravariant system of order two.

Equations (8) are the conditions that the multilinear forms

$$\sum_{r_1 \cdots r_m} Y_{r_1 \cdots r_m} dy_{r_1} \cdots dy_{r_m}, \quad \sum_{s_1 \cdots s_m} X_{s_1 \cdots s_m} dx_{s_1} \cdots dx_{s_m}$$

shall be transformable into one another. They are called *associated forms*.

If $X_{r_1 \cdots r_m}$ and $\Xi_{r_1 \cdots r_m}$ are typical terms of two covariant systems, then $X_{r_1 \cdots r_m} + \Xi_{r_1 \cdots r_m}$ is a term of a covariant system of the same order; in similar manner the quantities $X_{r_1 \cdots r_m} \Xi_{s_1 \cdots s_p}$ form a covariant system of order $m + p$. Analogous results are true of contravariant systems.

Of particular importance is the idea of *composition* of a covariant and a contravariant system. Thus, the two systems whose typical functions are

$$\sum_{s_1 \cdots s_p} \Xi^{(s_1 \cdots s_p)} X_{r_1 \cdots r_m s_1 \cdots s_p} \qquad \sum_{s_1 \cdots s_p} \Xi_{s_1 \cdots s_p} X^{(r_1 \cdots r_m s_1 \cdots s_p)}$$

can be shown to be covariant and contravariant respectively of order m.

The particular system $X^{(r_1 \cdots r_m)}$ defined by

$$\sum_{s_1 \cdots s_m} a^{(r_1 s_1)} \cdots a^{(r_m s_m)} X_{s_1 \cdots s_m}$$

is said to be the reciprocal system of $X_{r_1 \cdots r_m}$ with respect to the fundamental form F; and conversely.

There is one other fundamental idea namely *covariant and contravariant differentiation* with respect to a form F. The former is a generalization of equation (6) and the latter is analogous to it. It is significant that if $X_{r_1 \cdots r_m}$ be a term of a covariant system, the functions $X_{r_1 \cdots r_m hk}$ and $X_{r_1 \cdots r_m kh}$, obtained from the first by repeated covariant differentiation with respect to x_h and x_k, and inversely, are not equal in general.

Ricci and Levi-Civita have established by these means the following fundamental theorem:

In order to obtain all the absolute differential invariants of order μ involving the coefficients of F and functions $f_1, \cdots, f_r$ it suffices to determine the algebraic invariants of the following system of forms

(i) $$F,\ G_4, \cdots, G_{\mu-2};$$

(ii) the associated forms obtained from $f_1, \cdots, f_r$ with respect to F up to order μ.

Wright merely states the general theorem and illustrates the method of proof in the case of a binary form.

In 1884 Lie applied his theory of continuous groups to the determination of the differential invariants of a binary quadratic differential form. This necessitated the introduction of the idea of an "extended" group, consisting of the transformations operating not only upon the original variables x and y, but also upon such added variables, as the coefficients E, F, G of the quadratic form $Edx^2 + 2Fdxdy + Gdy^2$ and certain functions $f_1, \cdots, f_r$ and their derivatives with respect to x and y. This requires a knowledge of the variation of these quantities due to an infinitesimal transformation of the original group, written

$$x' = x + \xi(x, y)\delta t, \quad y = y + \eta(x, y)\delta t,$$

where ξ and η are arbitrary. Then if Ω is the infinitesmal operator of the group in the variables $x, y, E, F, G, f, f_x, f_y$,

$$\Omega \equiv \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \frac{\delta E}{\delta t}\frac{\partial}{\partial E} + \cdots + \frac{\delta f}{\delta t}\frac{\partial}{\partial f} + \cdots + \frac{\delta f_y}{\delta t}\frac{\partial}{\partial f_y},$$

and an invariant I must satisfy the condition $\Omega I = 0$, whatever be ξ and η. Wright calculates the quantities $\delta E, \cdots, \delta f_y$ directly and gives the complete system of linear equations in the above eight variables which I must satisfy. This is followed by an explanation of the case where the invariants involve functions f and their derivatives of order p, and an exposition of the general method of Forsyth for the calculation of the increments of a function.

The remainder of Chapter III deals with the determination of differential parameters for forms of rank zero, that is forms for which we can find functions u_λ satisfying the condition

(9) $$\sum_{r,s}^{n} a_{rs} dx_s dx_r = \sum_{\lambda}^{n} du_\lambda^2.$$

The variables entering are

(i) $u_1, u_2, \cdots, u_n$ and their total differentials;
(ii) n, or less, arbitrary functions $\phi^{(k)}$ and their derivatives;
(iii) the Jacobian J of the u's.

If we let F_m denote

$$\left\{\sum_i^n U_i \frac{\partial}{\partial u_i}\right\}^m \phi,$$

where the U's are a set of auxiliary variables, the function F_m is an algebraic form of order m whose coefficients are the mth derivatives of ϕ except for numerical multipliers. It is shown that the functions F are expressible as functions of the quantities a_{rs} and their derivatives. The final result is:

The most general invariant is a function of the quantities $d^r\phi^{(k)}$, the general algebraic invariants of the forms F and of ΣU^2, multiplied by some power of J.

In general more than n functions u_λ are necessary in order that the F may be given the form (9). If there are m $(>n)$, then F may be looked upon as a manifold of order n in the space of order m. In this case the problem of finding the invariants of F is that of determining which of the invariants of

$$\sum_1^m du_\lambda^2$$

are invariants of the manifold.

In several memoirs Maschke has developed a symbolic method for differential forms analogous to the Clebsch method for algebraic forms. In Chapter IV Wright gives an exposition of this method. The assumption is made that the quadratic form (1) can be expressed symbolically as $(df)^2$, where f is a symbolic function of the n variables. The derivatives of f, written $\partial f/\partial x_i \equiv f_i$, have no meaning separately, but $f_i f_x$ is to mean $a_{i\kappa}$. As in the case of algebraic forms, equivalent symbols must be introduced in number to the order of the a's appearing in an expression. If $\Phi^{(i)}$, for $i=1, \cdots, n$, are n invariant expressions of the quadratic form F, so that $\Phi^{(i)} = \Phi'^{(i)}$, where $\Phi'^{(i)}$ is $\Phi^{(i)}$ for the new set of variables y, it is shown that the function

$$(10) \qquad |a_{11}a_{22}\cdots a_{nn}|^{-\frac{1}{2}} \left|\frac{\partial\Phi^1}{\partial x_1}\frac{\partial\Phi^2}{\partial x_2}\cdots\frac{\partial\Phi^n}{\partial x_n}\right|$$

is an absolute invariant. This function, denoted by (Φ), is called an *invariantive constituent of the* form F. Any combination of these constituents which has a significance in terms of the a's is an invariant, and thus the invariants are formed.

By (f) is meant the invariantive constituent of equivalent symbols expressing the quadratic form; by (uf) the constituent in which u is any function of the n variables and the remainder of the Φ's in (Φ) are equivalent symbols. In particular, we have

$$(f)^2 = n!,\ (uf)^2 = (n-1)!\,\Delta_1 u,\ (uf)(vf) = (n-1)!\,\Delta(u, v),$$
$$((uf)f) = (n-1)!\,\Delta_2 u,$$

where $\Delta_1 u$, $\Delta(u, v)$ and $\Delta_2 u$ are the well-known differential parameters of F. In deriving the last of these identities use is made of covariant differentiation which is expressible thus

$$u^{(i\kappa)} = u_{i\kappa} - \frac{1}{(n-1!)^2} f_{i\kappa}(fa)(ua),$$

f and a being equivalent symbols for the fundamental form, and $u_{i\kappa}$ denoting

$$\frac{\partial^2 u}{\partial x_i \partial x_\kappa}.$$

In the reduction of the invariants to suitable form Maschke makes use of a large number of symbolic identities a few of which Wright gives. The Riemann symbols are expressible in an interesting form, namely

$$(i\kappa rs) = f_{ir} f^{(\kappa s)} - f_{is} f^{(\kappa r)},$$

the subscripts referring to ordinary differentiation of the symbolic function f and the superscripts to covariant differentiation. By means of these symbolic expressions for the convariant forms G_4 and G_5 are obtained. This chapter is very short, and is in fact a resumé with few details of Maschke's memoirs in the fourth volume of the *Transactions*.

The final chapter of the book, which constitutes about half of it, is devoted to applications. The first twelve pages deal with differential parameters of a single form for $n = 2$, the subject matter being similar to that which may be found in Volume III of Darboux's Leçons. The remainder sets forth some of

the applications of the absolute differential calculus. The functions used belong to covariant or contravariant systems, and the ordinary operation of differentiation is replaced by covariant or contravariant differentiation with respect to a fundamental form.

The author contributes a solution by these means of the problem of determining the most general triply orthogonal system of surfaces in a three dimensional manifold with a given quadratic form, and arrives at a differential equation of the third order which is an interesting generalization of the equation found by Darboux for euclidean space. The other applications are chosen from those given by Ricci and Levi-Civita. We shall call attention to several of them.

If $\lambda^{(1)}, \cdots, \lambda^{(n)}$ be n functions of $x_1, \cdots, x_n$ satisfying the condition

$$\sum_{ik} a_{ik}\lambda^{(i)}\lambda^{(k)} = 1, \tag{11}$$

the n equations

$$\frac{dx_i}{ds} = \lambda^{(i)}$$

define a family of curves, one through each point, which consequently may be said to be a congruence. It is readily seen that the *coefficients* $\lambda^{(i)}$ form a contravariant system. If λ_i denotes the *reciprocal* function of $\lambda^{(i)}$ then equation (11) may be written in the other forms

$$\sum_{i,\,k} a^{(ik)}\lambda_i\lambda_k = 1, \quad \sum_r \lambda^{(r)}\lambda_r = 1.$$

If we have n congruences such that the functions $\lambda_{h/r}$ and $\lambda_{k/r}$ (where h and k signalize the congruence and r the derivative) satisfy the conditions

$$\sum_r \lambda_h^{(r)}\lambda_{k/r} = \eta_{kh} \qquad (h, k = 1, \cdots, n),$$

where $\eta_{kk} = 0$ when $h \neq k$, and $\eta_{hh} = 1$, the set of congruences is said to form an *orthogonal ennuple.* An important theorem is that any covariant or contravariant system is expressible in terms of invariants and the coefficients of an orthogonal ennuple. In particular, the first covariant derivatives of the λ's may be expressed in terms of themselves and certain invariants γ_{hkl}, thus

$$\lambda_{h/s} = \sum_{k,\,l} \gamma_{hkl}\lambda_{k/r}\lambda_{l/s}.$$

These invariants satisfy the conditions

$$\gamma_{hhq} = 0, \quad \gamma_{hkq} + \gamma_{khq} = 0,$$

and so there are $\frac{1}{2}n^2(n-1)$ of them. They are a generalization of the rotations p, q, r, p_1, q_1, r_1 of the moving axes of Darboux, and like the latter they must satisfy certain partial differential equations of the first order.

The necessary and sufficient condition that a congruence $\lambda_{n|i}$ admit of a family of orthogonal surfaces is

$$\gamma_{nkh} = \gamma_{nhk} \quad (k, h = 1, \cdots, n-1).$$

A geodesic being defined as a curve for which the first variation of the integral of F in (1) is zero, the condition necessary and sufficient that a congruence $\lambda_{n|i}$ consist of geodesics is that $\gamma_{nin} = 0$, for $i = 1, \cdots, n$.

The absolute calculus may be applied also to dynamical problems. Consider any holonomic system and let its generalized coordinates be $x_1, \cdots, x_n$. The kinetic energy of the system, say T, is given by

$$2T = \sum_{r,s} a_{rs}\dot{x}_r\dot{x}_s,$$

where the dot denotes differentiation with respect to the time. If the system is subject to external forces depending only on the position, so that the work done in a small displacement is given by

$$\sum_r X_r dx_r,$$

the equations of motion are those of Lagrange

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_r}\right) - \frac{\partial T}{\partial x_r} = X_r. \tag{12}$$

The given dynamical problem may be regarded as solved when we know the most general values of the x's as functions of t which satisfy (12). These equations define a curve in n-space which is called a *trajectory* of the configuration. By choosing a congruence of these trajectories as one of the systems of an orthogonal ennuple the equations of Lagrange may be given an invariant form of great simplicity. In particular, it may be shown that if the external forces are all zero, the tra-

jectories are the geodesics of the manifold, and they are described with constant velocity. An important result is that the knowledge of an algebraic integral of the system (12) carries with it that of a homogeneous integral of the differential equations for geodesics in the manifold. The chapter closes with a discussion of homogeneous linear and quadratic integrals of the equations of geodesics; the determination in invariant form of the criteria that two dynamical systems have the same trajections; and the geodesic representation of one manifold upon another.

The subject matter is presented in an inspiring way, so that it seems very probable that the reader will turn to the papers of Ricci and Levi-Civita, as the author hopes. The proof reading has been well done and in every way the printer's work is satisfactory.

It is impossible to close this review without remarking the loss to American mathematics by the death of Mr. Wright. His brilliant record at Cambridge and his subsequent career in this country had won for him a high place in his field.

LUTHER PFAHLER EISENHART.

STURM'S GEOMETRISCHE VERWANDTSCHAFTEN.

Die Lehre von den geometrischen Verwandtschaften. Vierter Band: die nichtlinearen und die mehrdeutigen Verwandtschaften zweiter und dritter Stufe. By RUDOLF STURM. Leipzig and Berlin, Teubner, 1909. x + 486 pp.

As the subject matter of this fourth volume of Professor Sturm's extensive treatise on geometric relations is so different from that of the preceding ones,* but little analogy can be drawn with the methods already discussed. With the exception of one elementary treatise, mathematical literature did not include a book on birational transformations before the appearance of the present volume, although over five hundred memoirs have been devoted to the subject during the last two decades. The newness of the subject, the possibility of approaching it from different standpoints, and its applicability to so many other

*These volumes have been reviewed in the BULLETIN; volume 1 in vol. 15, p. 135; volume 2 in vol. 15, p. 252; volume 3 in vol. 16, p. 250.

disciplines all contribute to make the task of preparing a systematic treatise both important and difficult. Some aspects must be omitted altogether and others but briefly sketched. As to the wisdom of any particular choice of material there will doubtless be a wide diversity of opinion, for methods that are of immediate use for one purpose are of no value for another, although both serve to develop the same theory. The present volume proceeds synthetically, with a slight digression into more algebraic methods in discussing the general linear transformations of a complex variable. In most cases a brief outline only is given, the author constantly referring the reader to the original memoirs for the details; over three hundred such references are supplied. As in the preceding volumes, a glossary of technical terms is provided, the present one containing 74 terms.

In the earlier volumes a number of illustrations of Cremona transformations were met; in some cases the image of a straight line was actually found, and the general theorem that the genus of a curve is invariant under any birational transformation of the plane was proved (volume 1, no. 163). The present volume begins with the problem of finding the most general net of curves that can be imaged on the ∞^2 straight lines of the plane. Several theorems regarding the number of constants which determine a curve, the intersections of curves, and the law of equivalence of multiple points are stated as lemmas; from these criteria and the fact that the curves are rational it follows that each curve must be fixed by two linear conditions and that any two curves intersect in one point apart from the basis points of the net. The discussion occupies 20 pages, contains all the fundamental relations, includes a complete table of forms of nets up to order 10, and is admirably well presented. The next ten pages are devoted to illustrations, containing a veritable mine of suggestions for further research. The chapter on quadratic inversion is unnecessarily long (33 pages) and detailed; the various forms of the principal elements are not kept sharply separate. Conformal representation and transformation by reciprocal radii receive a great amount of attention. At the end of this chapter the complex variable is introduced and the preceding theory is developed analytically. The involutorial (simple) inversions are consistently distinguished from general non-periodic ones, but it is not shown that the latter can be obtained from the former by multiplication with a collineation.

Involutorial transformations of higher order are now discussed, largely following Doehlemann, but the Geiser transformation, fixed by a net of general cubic curves having seven distinct basis points, is also treated.

The first chapter closes with the theory of correspondence applied to ternary fields. The general formula for the number of coincidences is derived, then confirmed by numerous illustrations, taken from cases in which the number of coincidences is determined in another way.

The second chapter is devoted to correspondences on curves of genus one. Most of it is confined to the general plane cubic, then the space quartic of the first kind, the plane binodal quartic and the ruled quartic surface having two double directrix lines are considered. Each of these entities can be rationally mapped on any one of the others. The treatment is essentially that of Weyr, and is very complete, containing a full geometric theory of the properties of cubic curves and a well drawn picture of the configurations which define finite Cremona cycles, as well as of the G_{18} of collineations under which a non-singular, cubic curve remains invariant.

The problem of multiple correspondences between ternary fields is again taken up in an instructive chapter which considers in detail the (1, 2) case. The procedure is analogous to that of Paolis, but the latter is only an abstract of the wealth of material and particularly of new view points here developed. If the genus of the curves in the simple field which are the images of the straight lines in the double field is called the genus of the transformation, it is shown that we need only consider the two cases, genus zero and genus one. The basis points, the coincidence curve of the involution in the simple field, and the contact curves of the curve of branch points in the double field are exhaustively treated. A typical case is the Geiser transformation, already discussed in the preceding chapter. A later section considers (m, 1) and (2, 2) correspondences. Finally, a number of illustrations are added to show the usefulness and power of the ideas here developed. An immense field of uncultivated territory is opened by this discussion.

As a suitable transition from Cremona transformations in the plane to those in space, a chapter is introduced which treats of those surfaces which can be rationally mapped upon the plane. The general stereographic projection of a quadric surface is

followed by a much fuller discussion of the case of the cubic. Two general methods are considered, that of trilinear pencils of planes and that of the linear congruence whose directrices are two skew lines lying on the surface. This is followed by the Steiner surface, the cubic ruled surfaces, the quartic having a double conic and the quintic having a double space cubic. Finally, the depiction of a surface of order n which has an $n-2$-fold line and the Noether surface of order 4 are discussed. One interesting and important question in this connection is hardly touched upon, namely, to determine the nature of those transformations in space which go into Cremona transformations in the plane under which a plane curve remains invariant.

The birational transformations of space are introduced by making a system of surfaces having four homogeneous linear parameters collinear with the planes of space. The images of lines in one space and of planes in the other are shown to be of the same order, but those of planes in the first and of lines in the second are of an order independent of the first number, being fixed by the configuration of the principal elements. In order that the operations be reversible, the principal elements must absorb all but one of the intersection of any three surfaces of the system. The principal points, curves, and surfaces are brought in, together with the principal curves of the second kind, i. e., the locus of points whose images are all the same principal curve. The entire configuration is shown to constitute the Jacobian of the system. The images of lines in either system must be rational curves, and of planes must be homaloidal surfaces.

This short introduction is then amplified by studying various particular cases in detail, the first being a system of quadrics passing through four fixed points and touching a fixed plane at one of them. The system in the other space is composed of Steiner surfaces. When the quadrics have a line in common, the Steiner surfaces are replaced by cubic ruled surfaces. If the quadrics have a conic in common, the other system is also composed of quadrics. A particular form of this case is the transformation by reciprocal radii, which is discussed at some length. The second illustration is that of general cubic surfaces. It begins with Meusnier's theorem regarding the curvature of plane normal sections of a surface, in order to provide for an intersection of two cubics of the desired form. On account of the number of forms of the cubic surface this case

receives a long discussion. An extensive knowledge of the forms of cubic surfaces is presupposed, the author constantly referring to his own book on this subject.

The chapter on involutorial transformations is an enumeration of a number of the better known cases, such as generalized quadratic inversion, the polarity of a cubic surface from a point upon it, the harmonic conjugate of a point with regard to the points of intersection of a line through it and cutting a given cubic curve twice, and the pairs of associated points defining a bundle in a system of ∞^3 quadric surfaces passing through six fixed points, this last case being the Geiser transformation of space. The peculiarities of the fifteen lines joining the six points by twos and the cubic curve passing through all of them are discussed in detail. These lines are interesting as furnishing the first example of a principal curve of the second kind. The more general investigations of Montesano are not considered in the present volume.

The last chapter treats of multiple correspondences in space of three dimensions. It begins with the determination of the number of coincidences, and confirms the result by numerous simple illustrations. A short discussion is devoted to correspondences in line space; the general formula for the number of coincidences is derived and a simple illustration given. As in the case of the plane, the next section considers more in detail the (2, 1) correspondence first studied by Paölis. Most of the results are directly analogous in space to those obtained above for the plane, but to follow the proofs a knowledge of the author's treatise on line geometry is necessary. This interesting section is followed by two cases of (2, 2) quaternary correspondences, and a generalization of the duality defined by a linear complex (höhere Nullverwandtschaft). A short appendix completes the proofs of a few theorems in the preceding volumes and extends a few results to more general cases.

VIRGIL SNYDER.

NOTES.

THE Annual Register of the AMERICAN MATHEMATICAL SOCIETY is now in preparation and will be issued in January. Blanks for furnishing necessary information have been sent to the members. Early notice of any changes since the issue of the last Registerill wgreatly facilitate the work of the Secretary. The

Register is widely circulated and it is desirable that the information which it contains should be accurate and reliable.

THE concluding (October) number of volume 11 of the *Transactions of the American Mathematical Society* contains the following papers: "Conjugate line congruences contained in a bundle of quadric surfaces," by V. SNYDER; "On the fundamental number of the algebraic number-field $k(\sqrt[p]{m})$," by J. WESTLUND; "Volterra's integral equation of the second kind, with discontinuous kernel," by G. C. EVANS; "Ein Seitenstück zur Möbius'schen Geometrie der Kreisverwandtschaften," by H. BECK; "Vector interpretation of symbolic differential parameters," by L. INGOLD; "Surfaces with isothermal representation of their lines of curvature and the transformations (second memoir)," by L. P. EISENHART; "On the base of a relative number-field, with an application to the composition of fields," by G. E. WAHLIN; "The strain of a non-gravitating sphere of variable density," by L. M. HOSKINS.

THE concluding (October) number of volume 32 of the *American Journal of Mathematics* contains the following papers: "q-difference equations," by F. H. JACKSON; "On the relation between the sum-formulas of Hölder and Cesàro," by W. B. FORD; "Sur un exemple de fonction analytique partout continue," by D. POMPIEU; "Symmetric binary forms and involutions," by A. B. COBLE; "Systems of tautochrones in a general field of force," by A. W. REDDICK: "The general transformation theory of differential elements," by E. KASNER.

THE opening (October) number of volume 12 of the *Annals of Mathematics* contains the following papers: "The straight line solutions of the problem n bodies," by F. R. MOULTON; "On semi-analytic functions of two variables," by MAXIME BÔCHER; "Some theorems concerning systems of linear partial differential expressions," by W. J. BERRY; "Some circles associated with concylic points," by J. L. COOLIDGE; "On a method for the summation of series," by R. E. GLEASON.

AT the annual meeting of the Cambridge philosophical society on October 31 Mr. G. H. HARDY read a paper on "Fourier's double integral and the theory of divergent integrals."

THROUGH the gift of Mr. John Claflin the College of the City of New York has acquired the library of the late SIMON NEWCOMB, consisting of 4,000 books and 6,000 pamphlets.

ON September 4 more than a hundred participants founded

the Swiss mathematical society as a section of the Society of Swiss naturalists. The affairs of the society will be conducted by a council of three members, elected for a term of two years. The present council consists of Professor R. FUETER, of Basel, chairman, Professor H. FEHR, of Geneva, and Professor M. GROSSMAN, of Zürich, secretary. The first session was held September 6, at which the following papers were read: M. GROSSMAN, Zürich: "A geometric problem of photogrammetry." R. FUETER, Basel: "Principles of classification of algebraic numbers." F. PRÁSEL, Zürich: "Graphical methods in hydrotechnical problems." O. SPIESS, Basel: "Geometric considerations." D. MIRIMANOFF, Geneva: "The last theorem of Fermat." H. FEHR, Geneva: "Report of the international commission." E. MEISSNER, Zürich: "Concerning a surface not in tetrahedral position." F. RUDIO, Zürich: "Report on the publication of the works of Euler." R. LAEMNEL, Zürich: "Mathematics and biology."

THE firm of B. G. Teubner in Leipzig will celebrate the completion of its first century on March 3, 1911. A memorial address for the occasion is being prepared by Dr. F. SCHULZE; it will contain a biography of the founder of the firm, an account of its early activity in literature, and a detailed presentation of the scientific development of the publishing house since 1850.

AN excellent heliographic reproduction has been made of the photograph of Dirichlet mentioned in the BULLETIN, volume 15, page 318. Copies can be had for 3.35 marks by applying directly to Frl. Lotte Nelson, Darmstadt, Landskronstrasse 39.

UNIVERSITY OF PARIS. The following mathematical courses are announced for the semester beginning November 3, 1910:— By Professor G. DARBOUX: Infinitesimal geometry, cartography, two hours.— By Professor E. GOURSAT: Differential and integral calculus, elements of the theory of analytic functions, two hours.— By Professor E. BOREL: Integral functions, one hour.— By Professor P. APPELL and M. C. GUICHARD: General laws of equilibrium and motion, two hours.— By MM. CARTAN and BLUTEL: General mathematics, first part, two hours.— By Professor H. POINCARÉ: Cosmogonic hypotheses, two hours.— By Professor J. BOUSSINESQ: Mechanical theory of light, two hours.— By Professor G. KOENIGS: General theory of mechanisms, two hours.

Conferences will be conducted by Professor KOENIGS and MM. GUICHARD, CARTAN, BLUTEL, SERVANT, and ROUBAUDI.

In the Ecole Normale. By Professor J. TANNERY: Differential and integral calculus.— By Professor E. BOREL: Theory of functions.

MR. J. M. DODDS of Peterhouse, Mr. J. B. PEACE of Emmanuel College, and Mr. P. C. GAUL of Trinity College have been appointed examiners for Part I of the mathematical tripos at Cambridge for 1911.

ON the invitation of the committee of the Wolfskehl foundation, Professor H. A. LORENTZ, of the University of Leyden, delivered a course of lectures on the development of our ideas of the ether at the University of Göttingen during the week of October 24–29.

DR. E. FANTA has been appointed docent in mathematics and insurance at the German technical school at Brünn.

DR. H. MOHRMANN has been appointed docent in mathematics at the technical school of Karlsruhe.

DR. ROTHE has been appointed docent in mathematics at the technical school of Vienna.

DR. V. N. ROSEVARE has accepted the professorship of mathematics in the University of South Africa at Natal.

MR. H. BATEMAN, fellow of Trinity College, Cambridge, and reader in mathematical physics at the University of Manchester, has been appointed lecturer in mathematics at Bryn Mawr College.

AT Ohio Wesleyan University Professor G. N. ARMSTRONG has been promoted to a full professorship of mathematics.

DR. H. A. CONVERSE has been appointed head of the department of mathematics in the Baltimore Polytechnic Institute.

AT Harvard University Professor E. B. WILSON, of the Massachusetts Institute of Technology, has been appointed lecturer in mathematics for the second half of the present academic year. He will give a course on the "Applications of probabilities to theoretical physics." Dr. F. J. DOHMEN has been appointed instructor in mathematics.

AT the University of Colorado Professor S. EPSTEEN has been promoted to a full professorship of mathematics.

PROFESSOR R. B. MCCLENON, of Iowa College, has been promoted to an associate professorship of mathematics.

At Bryn Mawr College Dr. Isabel Maddison has been appointed to the office of recording dean.

At the College of the City of New York Dr. F. M. Pedersen has been promoted to an assistant professorship of mathematics.

Dr. Alexander Pell of the Armour Institute has been promoted from an assistant professorship to an associate professorship of mathematics.

Dr. H. W. Stager, of the University of California, has been appointed instructor in mathematics at Fresno Junior College.

Dr. E. W. Sheldon, of Yale University, has been appointed assistant professor of mathematics in the University of Alberta.

Mr. Meyer Gaba, of the University of Kansas, has been appointed instructor in mathematics in the school of mines of the University of Missouri, Rolla, Mo.

Dr. H. F. MacNeish, of Princeton University, has been appointed instructor in mathematics in the Sheffield Scientific School of Yale University.

Mr. H. Pritchard and Mr. J. Pitman have been appointed assistants in mathematics at Swarthmore College.

Mr. R. DuHadway has been appointed instructor in mathematics at the University of Iowa. Mr. F. Eaton has resigned his instructorship to accept a position in the new government college at Peking.

Mr. R. L. Cary has been appointed instructor in mathematics at Princeton University.

At the University of Georgia Dr. R. P. Stephens has been promoted from an adjunct professorship to an associate professorship of mathematics. Professor J. F. Messick, late of Randolph Macon College, has been appointed adjunct professor of mathematics.

Dr. J. K. Lamond, of Yale University, has been appointed instructor in mathematics at Wesleyan University.

At Brown University Dr. R. C. Archibald has resumed his academic work after a year's absence in Paris. Mr. F. W. Beal has been appointed instructor in mathematics. Mr. C. H. Currier has a year's leave of absence for study at Göttingen.

MISS IDA M. SCHOTTENFELS has been appointed head of the department of mathematics in Toledo University.

DR. A. E. HAYNES, professor of engineering mathematics in the University of Minnesota, has been granted a year's leave of absence. Professor Haynes has taught consecutively for thirty-five years, fifteen years at Hillsdale College, three years in the Michigan College of Mines, and seventeen years in the University of Minnesota.

PROFESSOR G. A. OSBORNE, of the Massachusetts Institute of Technology has been made professor emeritus.

PROFESSOR A. B. NELSON, of Central University, has retired from active service.

PROFESSOR J. LÜROTH, of the University of Freiburg, died September 14, at the age of 66 years.

PROFESSOR W. THOMÉ, of the University of Greifswald, died October 1, at the age of 69 years.

PROFESSOR P. WEINMEISTER, of the academy of forestry at Tharandt, died August 27, at the age of 62 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AHRENS (W.). Latein oder Deutsch? Die Sprachenfrage bei der Herausgabe der Werke Leonhard Eulers. Magdeburg, Peters, 1910. 8vo. 4 + 76 pp. M. 1.60

BACHMANN (P.). Zahlentheorie. Versuch einer Gesamtdarstellung dieser Wissenschaft in ihren Hauptteilen. In 6 Teilen. Teil I: Die Elemente der Zahlentheorie (1892). Anastatischer Neudruck. Leipzig, Teubner, 1910. M. 7.20

BARBETTE (E.). Les sommes de p^{mes} puissances distinctes égales à une p^{me} puissance. Avec une table des 5000 premiers nombres triangulaires. Liège, Gnusé, 1910. 4to. 154 pp. Fr. 12.50

BIANCHI (L.). Vorlesungen über Differentialgeometrie. Uebersetzt von M. Lukat. 2te vermehrte und verbesserte Auflage. Leipzig, Teubner, 1910. 8vo. 18 + 721 pp. Cloth. M. 24.60

BOGRAD (J.). Die Astroidenfläche. (Diss.) Bern, Suter, 1909. 4to. 40 pp. Fr. 1.00

BURALI-FORTI (C.) ET MARCOLONGO (R.). Eléments de calcul vectoriel avec de nombreuses applications à la géométrie, à la mécanique et à la physique mathématique, traduit de l'italien par S. Lattés. Paris, Hermann, 1910. 8vo. 229 pp. Fr. 8.00

BURGGRAF (G.). Die Γ-Funktion für komplexe Argumente. Schluss. (Progr.) Brünn, 1910. 8vo. 29 pp.

CARNOY (H.). Sur les travaux mathématiques de M. Ernest Lebon (Grands dictionnaires biographiques internationaux). Paris. Pouget, 1910. 16mo. 55 pp.

DINTZL (E.). Ueber einige Eigenschaften der Bernoullischen und analoger Zahlen. (Progr.) Wien, 1910. 8vo. 11 pp.

EGLI (M.). Beitrag zur Theorie unendlicher Systeme linearer Gleichungen mit unendlich vielei Unbekannten. (Diss.) Zürich, Leemann, 1910. 8vo. 61 pp.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Edition française dirigée par J. Molk. Tome I, volume 3: Théorie des nombres. Fascicule 3: Théorie arithmétique des formes, par Vahlen et Cahen. Propositions transcendantes de la théorie des nombres, par P. Bachmann, Hadamard et Maillet. Paris, Gauthier-Villars, 1910. 8vo. pp. 193–288. M. 3.00

ENESTRÖM (G.). Verzeichnis der Schriften Leonhard Euler's. Liefg. 1. Leipzig, Teubner, 1910. 8vo. 2 + 209 pp. M. 10.00

FERRARI (F.). Die geometrische Lösung der Aufgaben 3ten und 4ten Grades mittels des Lineals und einer festen Kurve 3ter Ordnung mit Rückkehrpunkt oder reellem Doppelpunkte. Münster, 1910. 8vo. 54 pp. M. 2.00

FLÜKIGER (H.). Die Flächenteilung des Dreiecks mit Hilfe der Hyperbel. (Diss.) Bern, Wyss, 1910. 8vo. 4 + 51 pp.

FREUND (E.). Die Brennpunktskreise der Kegelschnittslinien. (Progr.) Pilsen, 1909. 8vo. 10 pp.

GALLATLY (W.). The modern geometry of the triangle. London, Hodgson, 1910. 8vo. 70 pp. Sewed. 2s. 6d.

GABRIGOS (V.). Nuove curve (V) e loro proprietà. Roma, Accad. d. Lincei, 1910. 4to. 25 pp.

GEIGER (K.). Zwei Kurven zweiter Ordnung in zwei-, drei-, und vierpunktiger Berührung. Ergänzungen zum Programm für 1906–07. (Progr.) Landshut, 1910. 8vo. 27 pp.

GOURSAT (E.). Cours d'analyse mathématique. Seconde édition, entièrement refondue. Tome premier: Dérivées et différentielles. Intégrales définies. Développements en séries. Applications géométriques. Paris, Gauthier-Villars, 1910. 8vo. 6 + 647 pp. Fr. 20.00

HAAG. Familles de Lamé, composées de surfaces égales. (Thèse.) Paris, Gauthier-Villars, 1910. 4to. 89 pp.

HUBER (P.). Die Scyphoide, eine Kurve 4ter Ordnung. (Diss.) Bern, Dürrenmatt-Egger, 1910. 8vo. 4 + 52 pp.

IST Mathematik Hexerei? Von einem preussischen Schulmeister. Freiburg, Herder, 1910. 8vo. 4 + 68 pp. M. 1.20

KOPPE. Mathematische Modelle zum Selbstanfertigen. Nordhausen, Koppe, 1910. 9 plates + 11 pp. M. 1.50

KREIS (H.). Einige Anwendungen der Matricestheorie. Winterthur, Ziegler, 1910. 8vo. 65 pp.

LAGRANGE (COMTE). Le Travail Mathématique. 2e édition. Paris, Bressin, 1910. 8vo. 24 pp.

LATTÉS (S.). See BURALI-FORTI (C.).

LEBON (E.). Emile Picard. Biographie, bibliographie analytique des écrits. (Collection des Savants du jour.) Paris, Gauthier-Villars, 1910. 8vo. 8 + 80 pp. Fr. 7.00

LOEHRL (A.). Ueber conforme und äquilonge Transformationen im Raume. Ein Beitrag zur Geometrie der Kugeln. (Diss.) München, 1910.

LUKAT (M.). See BIANCHI (L.).

MALESSA (G.). Fokale Eigenschaften korrelativer Grundgebilde. (Diss.) Strassburg, 1909. 8vo. 35 pp.

MARCOLONGO (R.). See BURALI-FORTI (C.).

MIKAMI (Y.). Mathematical papers from the far east. Compiled and edited. (Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Heft 28.) Leipzig, Teubner, 1910. 8vo. 6 + 229 pp. M. 11.00

MILNE (W. P.). Homogeneous coordinates for use in colleges and schools. London, Arnold, 1910. 8vo. 176 pp. 5s.

MONTEL (P.). Leçons sur les séries de polynômes à une variable complexe. Paris, Gauthier-Villars, 1910. 8vo. 11 + 130 pp. Fr. 3.50

MORABITO (G.). Note sur une méthode élémentaire pour établir quelques formules de géométrie analytique plane. Liége, Pholien, 1910. 4to. 15 pp. Fr. 1.50

PASTOR (J. R.). Correspondencia de figuras elementales. Con aplicación al estudio de las figuras que engendran. Curso 1908–09. Madrid, Hernández, 1910.

PIESCHEL (F.). Ueber die Krümmung von Flächen in gewöhnlichen Flächenpunkten und die Krümmung der Mittelpunktsflächen $Lx^2 + My^2 + Nz^2 = 1$ insbesondere. (Progr.) Teschen, 1909. 8vo. 32 pp.

POLOSSUCHIN (O.). Ueber eine besondere Klasse von differentialen Funktionalgleichungen. (Diss.) Zürich, Aktien-Buchdr., 1910. 8vo. 52 pp.

REICHERT (G.). Ueber Zahlen und Zahlenverhältnisse. Berlin, Schmidt, 1910.

ROSENTHAL (A.). Untersuchungen über gleichflächige Polyeder. (Abhandlungen der kaiserlichen Leopoldinisch-Carolinischen Akademie der Naturforscher, Band 93.) Leipzig, Engelmann, 1910. 150 pp. M. 12.00

SCHWENDENWEIN (H.). Ein Satz aus der Zahlentheorie. (Progr.) Klagenfurt, 1910. 8vo. 22 pp.

TANNERY (J.). Introduction à la théorie des fonctions d'une variable. Tome II: Intégrales définies, développements en séries, langage géométrique, fonctions de variables imaginaires. Avec une note de M. Hadamard. Paris, Hermann, 1910. 8vo. 484 pp. Fr. 15.00

II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

AMIOT (A.). Trattato di geometria elementare. Nuova edizione. Parte II. Sesta impressione. Firenze, Le Monnier, 1910. 16mo. 221 pp. L. 2.00

AUTENHEIMER (F.). Elementarbuch der Differential- und Integralrechnung mit zahlreichen Anwendungen aus der Analysis, Geometrie, Mechanik und Physik. 6te verbesserte Auflage, bearbeitet von A. Donadt. Leipzig, Voigt, 1910. 8vo. 8 + 616 pp. M. 10.00

BAKER (W. M.) and BOURNE (A. A.). The student's arithmetic. London, Bell, 1910. 8vo. 336 pp. 2s. 6d.

BARR (T.). Practical mathematics for continuation classes. London, Blackie, 1910. 8vo. 240 pp. 2s.

BATES (E. L.) and CHARLESWORTH (F.). Practical mathematics and geometry. Specially arranged to meet the revised requirements of the board of education. London, Batsford, 1910. 8vo. 454 pp. 3s.

BAUSCHINGER (J.) und PETERS (J.). Logarithmisch-trigonometrische Tafeln, neu berechnet. Band 1: Tafel der 8-stelligen Logarithmen aller Zahlen von 1–200000. Leipzig, Engelmann, 1910. 8vo. 20 + 368 pp. Cloth. M. 18.50

——. Logarithmic-trigonometrical tables, with 8 decimal places. Containing the logarithms of all numbers from 1 to 200,000, and the logarithms of the trigonometrical functions for every sexagesimal second of the quadrant. 2 volumes. Volume 1: Table of logarithms to 8 places of all numbers from 1 to 200,000. London, Asher, 1910. 8vo. 20 + 368 pp. 18s. 6d.

BELLENGER (H.). See NEVEU (H.).

BERICHTE über den mathematischen Unterricht in Oesterreich, veranlasst durch die internationale mathematische Unterrichtskommission. Heft I, Inhalt: F. Bergmann, Der mathematische Unterricht an den Realschulen; K. Kraus, Der mathematische Unterricht an den Volks- und Bürgerschulen. Wien, 1910. 8vo. 85 pp. M. 1.80

BERSANO (G. B.). Algebra, per le scuole tecniche, per le commerciali di primo grado e per le scuole normali. Seconda edizione corretta. Torino, 1910. 8vo. 106 pp. L. 1.60

BOURLET (C.). Précis d'algèbre rédigé conformément aux programmes du 27 Juillet 1905. Corrigés des 573 exercises et problèmes avec la collaboration de J. Hulot. 3e édition, revue. Paris, Hachette, 1910. 16mo. 507 pp. Fr. 3.50

BOURNE (A. A.). See BAKER (W. M.).

BURALI-FORTI (C.) e RAMORINO (A.). Elementi di algebra, per le scuole medie inferiori. Terza edizione, intieramente rifatta. Torino, Gallizio, 1910. 8vo. 7 + 142 pp. L. 1.75

BURTON (P.). An anomaly in mathematics, as delivered in our textbooks. London, Sealy, 1910. 8vo. 64 pp. Sewed. 1s. 6d.

CHARLESWORTH (F.). See BATES (E. L.).

COLIN (C.) et GIROD (J.). Cours de géométrie, conformément aux programmes officiels du 26 Juillet, 1909. Paris, Alcan, 1910. 18mo. 8 + 180 pp. Fr. 1.80

COLLINS (J. V.). Practical algebra. First year course. New York, American Book, 1910. $0.85

DAVISON (C.). A class book of trigonometry. London, Cambridge University Press, 1910. 8vo. 208 pp. 3s.

DAVISSON (S. C.). College algebra. New York, Macmillan, 1910. 12mo. 243 pp. Half leather. $1.40

DONADT (A.). See AUTENHEIMER (F.).

DUPUIS (J.). Tables de logarithmes à cinq décimales. 30e édition. Paris, Hachette, 1910. 16mo. 4 + 230 pp. Fr. 2.00

ENGELHARDT (P.). Ueber die graphische und näherungsweise numerische Auflösung von Gleichungen 2ten und 3ten Grades mit einer Unbekannten im Unterrichte an der Mittelschule. (Progr.) Würzburg, 1910. 8vo. 33 pp.

ENRIQUES (F.) e AMALDI (U.). Nozioni di geometria, ad uso delle scuole complementari. Bologna, Zanichelli, 1910. 16mo. 7 + 195 pp. L. 1.80

GALDEANO (Z. G. DE). Nueva contribución á la enseñanza de la matemática. Con indicaciones de systematizacion matematica. Zaragoza, Casanañal, 1910.

GIROD (J.). See COLIN (C.).

GOULD (E. S.). Leçons élémentaires de calcul différentiel et intégral. Traduites et annotées par L. Pennequin. Paris, Thomas, 1910. 16mo. 162 pp. Fr. 4.00

GRAY (J. C.). Number by development. A method of number instruction. Primary. London, Lippincott, 1910. 8vo. 4s.

GUTZMER (A.). Bericht über die Tätigkeit des Deutschen Ausschusses für den mathematischen und naturwissenschaftlichen Unterricht im Jahre 1909. [Schriften des Deutschen Ausschusses für den mathematischen und naturwissenchaftlichen Unterricht. Heft 6.] Leipzig, Teubner, 1910.

HALL (H. S.) and KNIGHT (S. R.). Elementary algebra for schools (Chapters 28–44) with answers. London, Macmillan, 1910. 8vo. 310 pp. 2s. 6d.

HULOT (J.). See BOURLET (C.).

INTERMEDIATE applied mathematics papers from 1892 to 1910. (University tutorial series.) London, Clive, 1910. 8vo. 104 pp. Sewed. 2s. 6d.

INTERMEDIATE mathematics papers. From 1900 to 1910. London, Clive, 1910. 8vo. 92 pp. Sewed. 2s.

INTERNATIONAL Correspondence Schools. Tratado elemental de matemáticas y mecánica, arreglado especialmente para los estudiantes de las Escuelas internacionales por correspondencia. Scranton, 1910. 8vo. 287 pp.

KNIGHT (S. R.). See HALL (H. S.).

KOWALEWSKI (G.). Das Integral und seine geometrischen Anwendungen. (Forschung und Studium: Eine Sammlung mathematischer Monographien für Studierende, Heft 1.) Leipzig, Veit, 1910. 8vo. 3 + 86 pp. M. 3.00

LAISANT (C. A.). L'enseignement du calcul. Conseils aux instituteurs. Paris, Hachette, 1910. 8vo. 8 + 80 pp. Fr. 7.00

MCGIFFERT (J.). Mathematical short cuts. Troy, N. Y., Allen, 1910. 16mo. 31 pp. Paper. $0.25

MATRICULATION mathematics papers. Being the papers in elementary mathematics set at the matriculation examination of the University of London, from June, 1903, to June, 1910. With full solutions to the paper of June, 1910. London, Clive, 1910. 8vo. 120 pp. Sewed. 1s. 6d.

MATRICULATION model answers, heat, light, and sound. Being London University matriculation papers from June, 1907, to June, 1910. London, Clive, 1910. 8vo. Sewed. 2s.

MATRICULATION model answers. Mathematics, from September, 1907, to June, 1910. London, Clive, 1910. 8vo. 160 pp. Sewed. 2s.

MATRICULATION model answers, mechanics. The London University matriculation papers in mechanics, from September, 1903, to June, 1910. London, Clive, 1910. 8vo. 180 pp. Sewed. 2s.

NEVEU (H.) et BELLENGER (H.). Cours de géométrie théorique et pratique à l'usage des écoles primaires supérieures. 1ère année. Paris, Masson, 1910. 16mo. 275 pp. Fr. 2.00

——. Cours de géométrie théorique et pratique a l'usage des écoles primaires supérieures. 2e année. Paris, Masson, 1910. 16mo. 315 pp. Fr. 2.50

NOODT. Mathematische Unterrichtsbücher für höhere Mädchenschulen. Ausgabe B. 4 Hefte. Bielefeld, Velhagen, 1910. M. 5.50

OSBORNE (R. S.). Practical arithmetic. Examples and exercises. London, Wilson, 1910. 8vo. 11 + 270 pp. 2s. 6d.

PALATINI (F.). Aritmetica ed algebra, ad uso delle scuole medie superiori. Torino, Gallizio, 1910. 8vo. 7 + 355 pp. L. 3.50

PETERS (J.). See BAUSCHINGER (J.).

PICKEN (D. K.). The theory of elementary trigonometry. London, Whitcombe, 1910. 8vo. 2s. 6d.

RAMORINO (A.). See BURALI-FORTI (C.).

RIGGS (N. C.). Analytic geometry. New York, Macmillan, 1910. 12mo. 11 + 294 pp. Cloth. $1.60

ROTHROCK (D. A.). Elements of plane and spherical trigonometry. New York, Macmillan, 1910. 8vo. 11 + 147 + 99 pp. Cloth. $1.40

ROUBAUDI (C.). Cours de géométrie descriptive pour l'enseignement secondaire. Fascicule 2. Second cycle. 4e édition, revue. Paris, Corbeil, 1910. 8vo. 179 pp. Fr. 3.00

SMITH (D. E.). See WENTWORTH (G. A.).

SOLUTION of triangles. (Machinery reference series.) 2 volumes. Industrial Press, 1910. 8vo. Each, $0.25

SUPPANTSCHITSCH (R.). Lehrbuch der Arithmetik. Für die 4te und 5te Klasse der Gymnasien und Realgymnasien. Wien, Tempsky, 1910. 8vo. M. 4.20

VEGA (G.). Thesaurus logarithmorum completus. Vollständige Sammlung grösserer logarithmisch-trigonometrischer Tafeln. Neudruck. Mailand, 1910. 4to. 684 pp. M. 20.00

WENTWORTH (G. A.). Wentworth's plane geometry. Revised by G. Wentworth and D. E. Smith. Boston, Ginn, 1910. 12mo. 6 + 287 pp. Cloth. $0.80

III. APPLIED MATHEMATICS.

APPELL (P.) et DAUTHEVILLE (S.). Précis de mécanique rationnelle. Introduction à l'étude de la physique et de la mécanique appliquée. Paris, Gauthier-Villars, 1910. 8vo. 736 pp. Fr. 25.00

BARTLETT (F. W.). Engineering descriptive geometry. New York, Wiley, 1910. 8vo. 6 + 159 pp. $1.50

BEISCHER. Beispiele zur Differential- und Integralrechnung aus dem Gebiete der Physik. (Progr.) Ravensburg, 1909. 8vo. 44 pp.

BÉCOURT (L.). Le dessin technique. Cours professionnelle de dessin géométrique. Série B. Cahier I. 3e édition. Paris, Montrouge, 1910. 16mo. 16 pp. Fr. 1.00

BRECKENRIDGE (W. E.), MERSEREAU (S. F.) and MOORE (C. F.). Shop problems in mathematics. Boston, Ginn, 1910. $1.00

BYK (A.). Einführung in die kinetische Theorie der Gase. Bändchen I: Die idealen Gase. (Mathematisch-physikalische Schriften für Ingenieure und Studierende, herausgegeben von E. Jahnke; 10 I.) Leipzig, Teubner, 1910. 8vo. 5 + 102 pp. Cloth. M. 3.20

CALDARERA (F.). Memoria sul moto dei pianeti. Palermo, Virzi, 1910. 4to. 38 pp. L. 3.00

CHWOLSON (O. D.). Traité de physique. Edition française, traduction de E. Duvaux. Tome III, 2me fascicule: Thermodynamique générale. Fusion. Vaporisation. Paris, Hermann, 1910. 8vo. 744 pp. Fr. 11.00

COMBÉBIAC (G.). Les actions à distance. (Collection Scientia.) Paris, Gauthier-Villars, 1910. 8vo. 90 pp. Fr. 2.00

CZUBER (E.). Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung. Band II: Mathematische Statistik, mathematische Grundlagen der Lebensversicherung. 2te, sorgfältig durchgesehene und erweiterte Auflage. (Teubner's Sammlung von Lehrbüchern, Band IX, 2.) Leipzig, Teubner, 1910. 8vo. 10 + 470 pp. Cloth. M. 14.00

DAUTHEVILLE (S.). See APPELL (P.).

DEFOSSEZ (L.). Les cartes géographiques et leurs projections usuelles. Paris, Gauthier-Villars, 1910. 16mo. 118 pp. Fr. 2.75

DRYSDALE (C. V.). The foundations of alternating current theory. London, Van Nostrand, 1910. 8vo. 300 pp. $2.50

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band VI, Teil 2: Astronomie. Redigiert von K. Schwarzschild. Heft 3, pp. 335–462. Leipzig, Teubner, 1910. 8vo. M. 3.60

EWING (J. A.). The steam-engine and other heat engines. Third edition, revised and enlarged. London, Cambridge University Press, 1910. 8vo. 622 pp. 15s.

FISHLEIGH (W. T.). Problems in descriptive geometry for class and drawing room. A collection of over 900 definite problems, for students in engineering and technical schools. General problems, special cases, applications, with 85 practical figures. Ann Arbor, Fishleigh, 1910. 4to. 8 + 73 pp. $1.00

FRANKLIN (W. S.) and MACNUTT (B.). Mechanics and heat: a textbook for colleges and technical schools. London, Macmillan, 1910. 8vo. 7s. 6d.

GEIGEL (R.). Licht und Farbe. (Bücher der Naturwissenschaft herausgegeben von S. Günther.) Leipzig, Reclam, 1910. 220 pp. M. 1.00

GIORLI (E.). Il meccanico, per uso dei capitecnici, ecc. Seconda edizione ampliata. Milano, Hoepli, 1910. 24mo. 15 + 566 pp. L. 4.50

GOUARD (E.) et HIERNAUX (G.). Cours élémentaire de mécanique industrielle. Compléments de cinématique. Résistance des matériaux. Moteurs hydrauliques et machines à vapeur. Tome 2. Paris, Dunod, 1910. 8vo. 363 pp.

HARRIS (E. G.). Compressed air. Theory and computations. New York, McGraw-Hill, 1910. 8vo. 12 + 123 pp. $1.50

HARTMANN (F. M.). Elementary mechanics for engineering students. New York, Wiley, 1910. 16mo. 11 + 171 pp. $1.25

HECK (R. C. H.). Notes on elementary kinematics. New York, Van Nostrand, 1910. 8vo. 62 pp. Boards. $1.00

HIERNAUX (G.). See GOUARD (E.).

HOUGHTON (C. E.). The elements of mechanics of materials. New York, Van Nostrand, 1910. 8vo. 194 pp. Cloth. $2.00

JACKSON (C. S.) and ROBERTS (W. M.). A first dynamics. (Dent collection, edited by W. J. Greenstreet.) London, Dent, 1910. 16mo. 412 pp.

LECORNU (L.). Etude géométrique sur l'équilibre et la descente rectiligne de l'aéroplane. Paris, Dunod, 1909. 8vo. 15 pp.

MACNUTT (B.). See FRANKLIN (W. S.).

MAYOR (B.). Statique graphique des systèmes de l'espace. 1re partie, texte, 4 + 208 pp.; 2de partie, atlas, 2 pp. + 7 plates. Lausanne, Rouge, 1910. 8vo. Fr. 8.00

MERSEREAU (S. F.). See BRECKENRIDGE (W. E.).

MINKOWSKI (H.). Zwei Abhandlungen über die Grundgleichungen der Electrodynamik. Mit einem Einführungsworte von O. Blumenthal. (Fortschritte der mathematischen Wissenschaften in Monographien.) Leipzig, Teubner, 1910. 8vo. 82 pp. M. 2.40

MOORE (C. F.). See BRECKENRIDGE (W. E.).

OCAGNE (M. D'). Leçons sur la topométrie et la cubature des terrasses. Nouveau tirage comprenant des notions sommaires de nomographie, des notions élémentaires sur la probabilité des erreurs et une instruction sur l'usage de la règle à calcul. Paris, Gauthier-Villars, 1910. 8vo. 7 + 260 pp. Fr. 8.50

ORAM (H. J.). See SENNETT (R.).

ORLANDI (J.). Nouvelles tables tachéométriques pour calculer les distances réduites à l'horizon, les différences de niveau, les coordonnées rectangulaires et les courbes. 2e édition, radicalement modifiée. Sassari, Gallizzi, 1910. 8vo. 23 + 201 pp.

OSEEN (C. W.). Ein Satz über die Singularitäten, welche in der Bewegung einer reibenden und unzusammendrückbaren Flüssigkeit auftreten können. Berlin, Friedländer, 1910. 8vo. 17 pp. M. 1.00

RABOZÉE (H.). Cours de résistance des matériaux donné à l'Ecole d'application de l'artillerie et du génie de Belgique. Deuxième édition, par G. Leman. Gand, Meyer, 1910. 8vo. 30 + 993 pp. Fr. 30.00

REININGHAUS (F.). Kalenderreformvorschlag. Zürich, Füssli, 1910.

RENFER (H.). Lehrbuch der politischen Arithmetik. St. Gallen, Fehr, 1910. 8vo. 8 + 190 pp. Fr. 5.75

RICHALD (J.). Cours de constructions à l'Ecole spéciale du génie civil de l'Université de Gand. Ouvrage d'art: troisième partie: barrage et écluses. Premier fascicule: barrages de réservoirs et barrages en rivières. Bruxelles, Ramlot, 1910. 4to. 181 pp. Fr. 7.00

ROBERTS (W. M.). See JACKSON (C. S.).

SENNETT (R.) and ORAM (H. J.). The marine steam engine. 10th edition. London, Longmans, 1910. 8vo. 510 pp. 21s.

SOUTHALL (J. P. C.). The principles and methods of geometrical optics, especially as applied to the theory of optical instruments. New York, Macmillan, 1910. 8vo. 23 + 626 pp. Cloth. $5.50

SPANGLER (H. W.). Applied thermodynamics. Philadelphia, McVey, 1910. 8vo. 160 pp. Cloth. $2.50

TRUMBULL (L. W.). A manual of underground surveying. New York, McGraw-Hill, 1910. 8vo. 10 + 251 pp. $3.00

WALLACE (J.). Logarithmic land measurements. London, Spon, 1910. 8vo. 6s.

WEBER (H.). Die partiellen Differential-Gleichungen der mathematischen Physik. Nach Riemann's Vorlesungen in 5 Aufl. bearbeitet. Band I. Braunschweig, Vieweg, 1910. 8vo. 18 + 528 pp. M. 13.60